On the Vertex Separation of Cactus Graphs and Maximal Outerplanar Graphs

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Abstract

The vertex separation of an undirected graph G, denoted by $vs(G)$, is the minimum over all linear layouts of G of the maximum number of vertices that are left of, or coincide with, any vertex u in the layout. A linear layout of G is any arrangement of the vertices of G in a line. The corresponding computational problem VERTEX SEPARATION (VS) is $\mathcal{NP}$-complete on general graphs and remains $\mathcal{NP}$-complete on planar graphs with maximum degree three. It has numerous applications and is equivalent to several other problems, the most famous among them being PATHWIDTH and NODE SEARCHING. The latter problem is to search under certain search rules the graph for an invisible, infinitely fast fugitive that hides on the edges.

Our goal is to develop fast and practical algorithms for the vertex separation of restricted graph classes, namely cactus graphs (cacti) and maximal outerplanar graphs (mops). Such fast and practical algorithms are known for only few graph classes, e.g. trees [2] and unicyclic graphs [1]. The first algorithm is based on a major theoretic result: a theorem that says for any tree T, $vs(T) \leq k$ iff for any vertex $u \in T$ at most two subtrees T_1, T_2 induced by u have vertex separation k and all other induced subtrees have vertex separation $\leq k-1$. The second algorithm does not have such a theoretical foundation but is rather based on exploring quite a large number of possible cases and subcases that arise when considering the trees attached to the cycle of a unicyclic graph.

Our investigation of VS on cacti takes the systematic approach of the solution of VS on trees. We prove a theorem that says how the vertex separation of a cactus depends on:

- the vertex separation of certain subcacti relative to each vertex, and
- the stretchability with respect to a couple of degree two vertices u and v of certain subcacti relative to each cycle.

The stretchability with respect to u and v is best explained using the searching terminology. The question is whether we can search the graph in such a way that the search starts at u and finishes at v, or *vice versa*, using at most a certain number of searchers. In order to construct an algorithm we further need to show how the new parameter, namely the stretchability, depends on the vertex separations and stretchabilities of subgraphs. That turns out to be a formidable task. We show that the generalisation of the *ad hoc* approach from [1] does not work because the stretchability with respect to a couple of vertices may reduce to stretchability with respect to two couples. In its turn, it may reduce to stretchability with respect to three couples, *etc.*

We therefore take the following approach: take STRETCHABILITY (with respect to arbitrarily many vertex couples) as our primary problem. We have shown [3] that stretchability is left-right independent, that is, it does not matter which vertex set is associated with the left direction and which, with the right one (in the terms of searching, it does not

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matter whether we start the search at some $U \subset V(G)$ and finish on some $W \subset V(G)$, or *vice versa*). The original problem VS is clearly a special case of STRETCHABILITY (STR). So we consider STR on special kind of cacti called *boundaried cacti* and we prove a theorem that, with respect to each vertex and cycles, reduces the stretchability to the stretchabilities of certain subgraphs that are boundaried cacti. Based on that, we construct an algorithm that computes the stretchability of a boundaried cactus. The algorithm, its verification and time complexity analysis are quite involved and are still work in progress.

The other graph class, namely the mops, seems to be even harder with respect to VS. We have proven a theorem that reduces VS to stretchabilities of submops relative to each face. The stretchability of a mop with respect to two vertices from some outer edge is quite hard to grasp in terms of a finite number of parameters of certain subgraphs. As of now, we can only show that the said stretchability may reduce to the vertex separation of a subgraph that is, unlike the mops, not 2-connected. That means that to solve VS on mops is not any easier than to solve VS on graphs with cut vertices whose blocks are mops. Thus we reject the hope that the rigid structure of the mops provides any ease with respect to the solution of VS as opposed to more general graph classes whose treewidth is ≤ 2.

References

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