Cell Structures

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Theorem

X compact metric space $\Longrightarrow \exists\, f : C \twoheadrightarrow X$ *(onto).*

Proof.

$$
\begin{array}{ccc}
\mathcal{U}_0 = \{C\} & \xrightarrow{f_0} & \mathcal{V}_0 = \{X\} \\
\uparrow i_{0,1} & & \uparrow j_{0,1} \\
\mathcal{U}_1 = \{U_{1,1}, \ldots, U_{1,n_1}\} & \xrightarrow{f_1} & \mathcal{V}_1 = \{V_{1,1}, \ldots, V_{1,n_1}\}, \ \text{mesh } < 1 \\
\text{disjoint clopen cover} & & \text{closed cover} \\
\uparrow i_{1,2} & & \uparrow j_{1,2} \\
\mathcal{U}_2 = \mathcal{U}_{2,1} \cup \cdots \cup \mathcal{U}_{2,n_1} & \xrightarrow{f_2} & \mathcal{V}_2 = \mathcal{V}_{2,1} \cup \cdots \cup V_{2,n_1}, \ \text{mesh } < 1/2 \\
\mathcal{U}_{2,i} \text{ disjoint cover} & & \mathcal{V}_{2,i} \text{ closed cover of } V_{1,i} \\
\text{of } U_{1,i} \text{ by clopen sets} & & \\
\uparrow i_{2,3} & & \uparrow j_{2,3} \\
\cdots\cdots & \cdots\cdots & \cdots\cdots \\
C \approx \varprojlim(\mathcal{U}_i, i_{j,k}) & \xrightarrow{\tilde{f}=(f_0, f_1, \ldots)} & \mathcal{V}_\infty = \varprojlim(\mathcal{V}_i, i_{j,k})
\end{array}
$$

Define $\sim$ on $\mathcal{V}_\infty$ by $(x_0, x_1, \ldots) \sim (y_0, y_1, \ldots)$ iff $x_i \cap y_i \neq \varnothing$ for all i

$$\pi \colon \mathcal{V}_\infty \to \mathcal{V}_\infty / \sim \ \approx X \qquad f = \pi \circ \tilde{f}$$

Cell structures are inverse systems of discrete spaces with combinatorial "nearness" relations.

Natural quotients (induced by the nearness relations) of the inverse limits yield topological spaces.

"Maps" between cell structures yield mappings between the corresponding quotient spaces.

Freudenthal (1937). Every compact metric space X admits a polyhedral inverse sequence with surjective bonding maps whose inverse limit is X.

Definition

Let Σ be a class of compact, connected polyhedra. Then the class $[\Sigma]$ of $\Sigma-$ like continua consists of all continua X such that there exist continuous mappings of X with arbitrarily small fibers onto some members of Σ.

Let (Σ) be class of continua X such that $X \approx \varprojlim(P_n, p_{\alpha,\alpha'})$ where $P_\alpha \in \Sigma$ and $p_{\alpha,\alpha'}$ are continuous surjections.

Note: $\Sigma \subset (\Sigma) \subset [\Sigma]$.

Mardesic-Segal (1963). Let Σ be a class of compact, connected polyhedra.

$$\text{metric} \cap [\Sigma] = \text{metric} \cap (\Sigma)$$

eg.

1) $\Sigma = \{[0,1]\}$, $\sin -1/x$ curve is arc-like i.e. $\sin -1/x$ curve $\in [\Sigma]$.
2) $\Sigma = \{$connected, finite, polyhedra of dim $\leq n\}$. Then metric $\cap[\Sigma] = \{$metric continua of dim $\leq n\}$.
3) **Pasynkov, Mardesic (1959)** - non-metric case problems - dim $\exists\, X$ non-metric, $X \in [\{[0,1]\}]$ but $X \notin (\{[0,1]\})$.

Mardesic (1963). Let Σ be a class of compact, connected polyhedra. $[\Sigma]$ = class of $\varprojlim(X_\alpha, p_{\alpha,\alpha'})$ where X_α are metric Σ-like continua, $\dim(X_\alpha) \leq \dim(X)$ and all $p_{\alpha,\alpha'}$ are continuous surjections.

Corollary

Every continuum X in $[\Sigma]$ is the limit of a double iterated inverse system of polyhedra in Σ with $\dim \leq \dim(X)$.

For mappings inverse limits are more problematic.

Mioduszewski (1963).

If X, Y metric compacta and $f\colon X = \varprojlim(P_i, p_{i,j}) \longrightarrow Y = \varprojlim(Q_i, q_{i,j})$ then for each sequence $\{\varepsilon_i\}$ of positive numbers $\epsilon_i \to 0$ there exists

$$\begin{array}{ccccccccccc}
P_{m_1} & \xleftarrow{p_{m_1,m_2}} & P_{m_2} & \xleftarrow{p_{m_2,m_3}} & P_{m_3} & \longleftarrow & \dots & \longleftarrow & P_{m_k} & \xleftarrow{p_{m_k,m_{k+1}}} & \dots \\
{\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow & & {\scriptstyle f_3}\downarrow & & \dots & & {\scriptstyle f_k}\downarrow & & \dots \\
Q_{n_1} & \xleftarrow{q_{n_1,n_2}} & Q_{n_2} & \xleftarrow{q_{n_2,n_3}} & Q_{n_3} & \longleftarrow & \dots & \longleftarrow & Q_{n_k} & \xleftarrow{q_{n_k,n_{k+1}}} & \dots
\end{array}$$

where $m_1 < m_2 < m_3 < \dots\ , \quad n_1 < n_2 < n_3 < \dots$

and each diagram

$$\begin{array}{ccccc}
 & & P_{m_k} & \longleftarrow & P_{m_r} \\
 & & \downarrow & & \downarrow \\
Q_{n_i} & \longleftarrow & Q_{n_k} & \longleftarrow & Q_{n_r}
\end{array}$$

is ε_k commutative for $i < k < r$.

Mardesic (1963). Let Σ and J be classes of connected polyhedra. Given $X \in [\Sigma]$, $Y \in [J]$ and $f\colon X \to Y$ a continuous surjection, then there exist inverse systems $X' = \varprojlim(X_\alpha, p_{\alpha,\alpha'})$ and $Y' = \varprojlim(Y_\beta, p_{\beta,\beta'})$ of metric continua in $[\Sigma]$ (resp. $[J]$) with surjective bonding maps and homeomorphisms h and k so

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \approx h\downarrow & \hookrightarrow & \approx k\downarrow \\ X' = \varprojlim(X_\alpha, p_{\alpha,\alpha'}) & \xrightarrow{f_\beta} & Y' = \varprojlim(Y_\beta, p_{\beta,\beta'}) \end{array}$$

Note : X_α and Y_β can not be taken to be polyhedra even if X and Y are metric.

Mardesic (1981). Resolutions - inverse systems with additional conditions to study noncompact cases.

Mardesic - Watanabe (1989). Approximate inverse systems and approximate resolutions to obtain rather arbitrary spaces and mappings and filling in the deficiencies indicated above.

Definition

- *Graph* is ordered pair (G, r).
- G is discrete set and r is reflexive and symmetric relation on G.
- *Cells* are points of G.
- a, b are adjacent if $(a, b) \in r$.
- $\mathrm{st}(a) = \{b \in G \mid (a, b) \in r\}$.

Let $\{(G_i, r_i) \mid i = 1, 2, \dots\}$ be graphs and
$g_{i,j} \colon G_j \to G_i$ be functions for $j \geq i$ satisfying

i) $g_{i,i} =$ identity on G_i

ii) $g_{i,k} = g_{i,j} \circ g_{j,k}$ for $i < j < k$ and

iii) $(a, b) \in r_{i+1} \implies (g_{i,i+1}(a), g_{i,i+1}(b)) \in r_i$

$$G_1 \xleftarrow{g_{1,2}} G_2 \xleftarrow{g_{2,3}} G_3 \xleftarrow{g_{3,4}} \dots$$

If $a \in G_i$, say $\deg(a) = i$.
Let $\Pi = \Pi G_i$ topological product.
Π is complete, 0-dimensional, metric space.
$G_\infty = \varprojlim(G_i, g_{i,j})$.
If $x = (x(1), x(2), \dots) \in \Pi$ then $x \in G_\infty$ iff $g_{i,i+1}(x(i+1)) = x(i)$ for each i.
G_∞ is set of *threads*.

Let $g_i \colon G_\infty \to G_i$ be ith coordinate projection.
If $a \in G_i$ let $< a >= \{x \in G_\infty \mid x(i) = a\} = g_i^{-1}(a)$.
$\{< x(i) >: x \in G_\infty, i = 1, 2, \dots\}$ is basis of open and closed sets for G_∞.

Proposition

G_∞ is closed in Π hence G_∞ is topologically complete 0-dimensional metric space eg. if each G_i is countable, G_∞ is closed subset of irrationals.

Definition

Set $x \sim y$ in G_∞ if $(x(i), y(i)) \in r_i$ for each i.
$\sim$ is reflexive and symmetric relation on G_∞.

Proposition

$\sim$ is closed in $G_\infty \times G_\infty$.

Proof.

$\sim = \bigcap R_i$ where $R_i = \{(x, y) \in G_\infty \times G_\infty \mid (x(i), y(i)) \in r_i\}$.
R_i is closed since r_i is closed in discrete space $G_i \times G_i$. □

Definition

Cells $a \in G_m$ and $b \in G_n$ are *close* if $(g_{k,m}(a), g_{k,n}(b)) \in r_k$ for $k = \min\{m, n\}$.

Cauchy Sequence is a sequence of cells $\{u(j)\}$ in $\cup G_i$ such that

iv) $\lim \deg(u(j)) = \infty$ and

v) $u(i)$ and $u(j)$ are close for all i and j sufficiently large.

Cauchy sequence $\{u(i)\}$ *converges* to thread $x \in G_\infty$ if $x(i)$ and $u(j)$ are close for all i and sufficiently large j.

Note. A Cauchy sequence may converge to different threads.

Definition

A *cell structure* is an inverse sequence of graphs satisfying

vi) $\forall$ thread $x \in G_\infty$, $\forall i$, $\exists j \geq i$ s.t. $g_{i,j}(\mathrm{st}^2_{r_j}(x(j))) \subset \mathrm{st}_{r_j}(x(i))$

vii) $\forall$ thread x, $\forall i$, $\exists j \geq i$ s.t $g_{i,j}(\mathrm{st_{r_j}(x(j))})$ is finite.

viii) each Cauchy sequence of cells converges

eg. $G_i = \{1, 2, \dots\}$, $r_i = \Delta \cup \{(k, l) \mid k, l \geq i\}$ and $g_{i,j} =$ identity.

$\{((G_i, r_i), g_{i,j})\}$ satisfies vi) and vii) but not viii).
$x = \{x(i) = i\}$ is Cauchy but does not converge.
Note. In general if vi) and vii) are satisfied and each set of mutually adjacent cells is finite then viii) is satisfied.

Let $(*) = \{((G_i, r_i), g_{i,j})\}$ be a cell structure. $\sim$ is transitive by vi).

Definition

For $x \in G_\infty$ let $[x] = \{y \in G_\infty \mid x \sim y\}$. By vii) $[x]$ is compact. Define $\pi\colon G_\infty \to G_\infty/\sim = G^*$. π is perfect mapping.
G^*, the *space determined by the cell structure*, is topologically complete metrizable space.

Proposition

If $(*)$ *is cell structure then*
$\{G^* \setminus \pi(G_i \setminus < A >) \mid A \subset G_i, i = 1, 2, \ldots\}$ *is basis for topology on* G^**.*

eg. $G_i = \{p10^{-i} \mid p \textit{ integer}\}$
$(x, y) \in r_i$ *iff* $|x - y| \le 10^{-i}$.
$g_{i,j}$ *an order preserving retraction.*
$G* \equiv \mathbb{R}$.

Theorem (1)

Each complete metric space is homeomorphic to a space determined by some cell structure.

Proof.

X paracompact $\longrightarrow \exists\{\mathcal{U}_i\}$ sequence of locally finite open covers with mesh $\mathcal{U}_i < 1/i$ and $\mathcal{U}_{i+1}$ closed star refines $\mathcal{U}_i$.
Define $g_{i,i+1}\colon \mathcal{U}_{i+1} \to \mathcal{U}_i$ by $g_{i,i+1}(U) \supset \mathrm{cl}(\mathrm{st}(U,\mathcal{U}_{i+1}))$.
Define $r_i = \{U \times V \mid U, V \in \mathcal{U}_i$ and $U \cap V \neq \varnothing\}$.
$\{((\mathcal{U}_i, r_i), g_{i,j})\}$ is a cell structure.
vi) follows from closed star refinement of $\mathcal{U}_{i+1}$ in $\mathcal{U}_i$.
vii) is local finiteness of covers
viii) if $\{u(i)\}$ is Cauchy sequence in $\bigcup \mathcal{U}_i$, most pairs $(u(i), u(j))$ intersect so form a Cauchy sequence in X converging to a point x in X. Choose inductively v(i) in $\mathcal{U}_i$ so $x \in v(i)$ and $v = (v(1), v(2), \dots) \subset G_\infty$. Then $\{u(i)\}$ converges to v.
Define $\varphi\colon \mathcal{U}^* \to X$ by $\varphi([x]) = \cap x(i)$. □

Cell maps. Let

$$*) \quad G_0 \xleftarrow{g_{0,1}} G_1 \xleftarrow{g_{1,2}} G_2 \ldots$$

$$*') \quad H_0 \xleftarrow{h_{0,1}} H_1 \xleftarrow{h_{1,2}} H_2 \ldots$$

be cell structures where G_0 and H_0 are singletons and where r_i and r_i' are the symmetric and reflexive relations on G and H respectively. Let π (resp. π') be the quotient maps of G_∞ onto G^* (resp. H_∞ onto H^*). A function $f\colon \bigcup G_i \to \bigcup H_i$ is called a *cell map* of *) to *') if f takes close cells to close cells and Cauchy sequences to Cauchy sequences.

Proposition

The composition of cell maps is a cell map.

Theorem (2)

*Let $f \colon \bigcup G_i \to \bigcup H_i$ be a cell map of cell structure *) to cell structure *'). Then f induces a continuous function $f^* \colon G^* \to H^*$ defined as follows. For x, a thread in G_∞ $\quad f^*(\pi(x)) = \pi^{'}(y)$ where y is a thread in H_∞ such that $f(x)$ converges to y.*

Proof.

By applying vi) twice get f^* is well defined. Continuity proved by contradiction using : (Error estimation). If $a \in G_i$ then $f^*(\pi(< a >)) \subset \pi^{'}(\mathrm{st}_{r_j'}(f(a)))$ where $j = \deg(f(a))$ □

Theorem (3)

Let (X, d) and (Y, ρ) be complete metric spaces. Let $\{\mathcal{U}_i\}$ and $\{\mathcal{V}_i\}$ sequences of locally finite open covers of X and Y respectively such that $\mathrm{mesh}(\mathcal{U}_i) < 1/i$, $\mathrm{mesh}(\mathcal{V}_i) < 1/i$, $\mathcal{U}_{i+1}$ closed star refines $\mathcal{U}_i$ and $\mathcal{V}_{i+1}$ closed star refines $\mathcal{V}_i$ for each i and $\mathcal{U}_0 = \{X\}$ and $\mathcal{V}_0 = \{Y\}$. Then each continuous function $F\colon X \to Y$ is induced by a cell map of $\bigcup \mathcal{U}_i$ to $\bigcup \mathcal{V}_i$.

Proof.

As in proof of Theorem 1 the sequences of covers $\{\mathcal{U}_i\}$ and $\{\mathcal{V}_i\}$ define cell structures $\mathcal{U}$ and $\mathcal{V}$ respectively. Let $F\colon X \to Y$ be a continuous function. Define a cell map $f\colon \bigcup \mathcal{U}_i \to \bigcup \mathcal{V}_i$ by setting for $U \in \mathcal{U}_i$, $f(U) \in \mathcal{V}_j$ such that $F(\mathrm{st}^3_{r_i}(U)) \subset f(U)$ where j is as large as possible if such j exists otherwise choose any $j > i$. If we identify $\mathcal{U}^*$ with X and $\mathcal{V}^*$ with Y then $f^* = F$. □

Some classes of spaces. Let $\{((G_i, r_i), g_{i,j})\}$ be a cell structure.

1) If each $r_i = \Delta$ then G^* yield all topologically complete 0-dimensional metric spaces.
2) If graphs G_i are finite then G^* yield all compact metric spaces.
3) If graphs G_i are finite and connected then G^* yield all metric continua.
4) If all G_i are finite trees then G^* yield all metric treelike continua.
5) If all G_i have each mutually adjacent set of cells of cardinality $\leq n+1$ then G^* yield all at most n-dimensional complete metric spaces.

Proof of 5).

If each set of mutually adjacent cells in each G_i has cardinality $\leq n+1$ then $\mathrm{card}([x]) \leq n+1$ so by Hurewicz theorem $\dim(G^*) \leq n$. If $\dim(X) \leq n$ then by Ostrand theorem there exist open covers $\mathcal{U}_i = \mathcal{V}_{i,1} \cup \cdots \cup \mathcal{V}_{i,n+1}$ of mesh $< 1/i$ such that each $\mathcal{V}_{i,j}$ is discrete. Hence, in each $\mathcal{U}_i$ each collection of mutually adjacent elements has cardinality $\leq n+1$. □

THANK YOU