Homotopy Groups of Continua as Topological Group

Shapes, quotients, and a clash of two categories

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- If X is locally complicated $\pi_n(X, p)$ often 'wants' to have an interesting topology so that the topology of $\pi_n(X, p)$ is an invariant of X itself.
- In particular if $\pi_n(X, p)$ is **isomorphic** to $\pi_n(Y, q)$ we can hope to distinguish X and Y by asking if $\pi_n(X, p)$ is **homeomorphic or not** to $\pi_n(Y, q)$.

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- We will make these answers more precise soon

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- Planar and other low dimensional Peano continua illustrate the meaning and usefulness of the 3 defintions/tools.
- $\pi_n(X, p)$ with quotient topology accentuates a fundamental shortcoming in the general definition of product topology of $G \times H$, making the case for example, for the relevance and utility of the category of sequential spaces SEQ.

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- Every pseudometric space generates a canonical metric **(Kolmogorov)** quotient, $x\tilde{}y$ iff $D(x, y) = 0$

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- Glue together countably many copies of $[0, 1]$ at 0, yields distinct T_2 quotients.

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- impose the **pseudo-metric quotient** on $\pi_n^{pseudometric}(X,p)$.

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- (If don't know much shape theory, embed $X \subset l_2$, let U_m be the union of finitely many $\frac{1}{2^m}$ open balls covering X, arrange $U_{n+1} \subset U_n$, ϕ is induced by $j : X \to \lim_{\leftarrow} U_n$ with inclusion bonding maps).

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- The continuous isomorphism $id : \pi_n^{quotient}(X, p) \to \pi_n^{pseudometric}(X, p)$
- might NOT be a homeomorphism ([F] 2005 AGT)
- In fact $\pi_1(HE, p)$ is **not** a topological group in TOP.

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- Moral: If X is a Peano continuum the image of $\pi_1(X,p)$ in the first shape group can be understood intrinsically and geometrically without reference to open covers of X

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- It is locally compact but the topology of $\pi_1(X,p)$ depends on the metric of X
- This is why, to get a nice theory, it is helpful to assume X is a compact metric space or **continuum**