Unicoherence of symmetric products and related spaces

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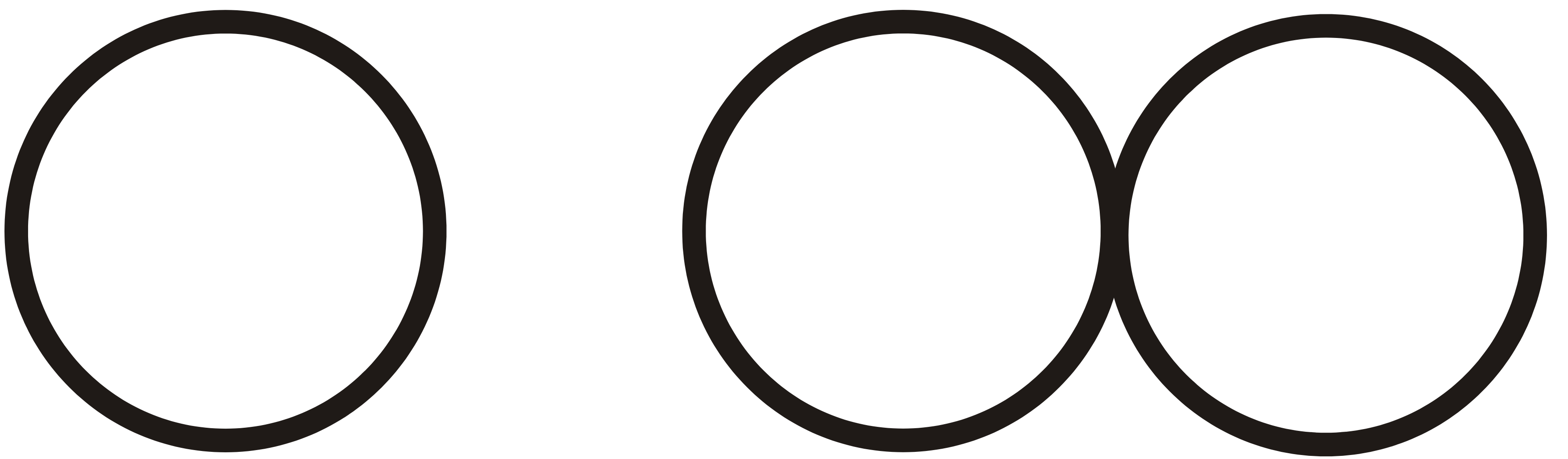
A **continuum** is a compact connected metric space with more than one point.

A continuum X is **unicoherent** provided that $A \cap B$ is connected whenever A and B are subcontinua of X such that $X = A \cup B$.

The **multicoherence degree** of a continuum X is defined as

$r(X) = \sup\{\text{number of components of } A \cap B$ whenever A and B are subcontinua of X such that $X = A \cup B\} - 1$

-A continuum X is unicoherent if and only if $r(X) = 0$.



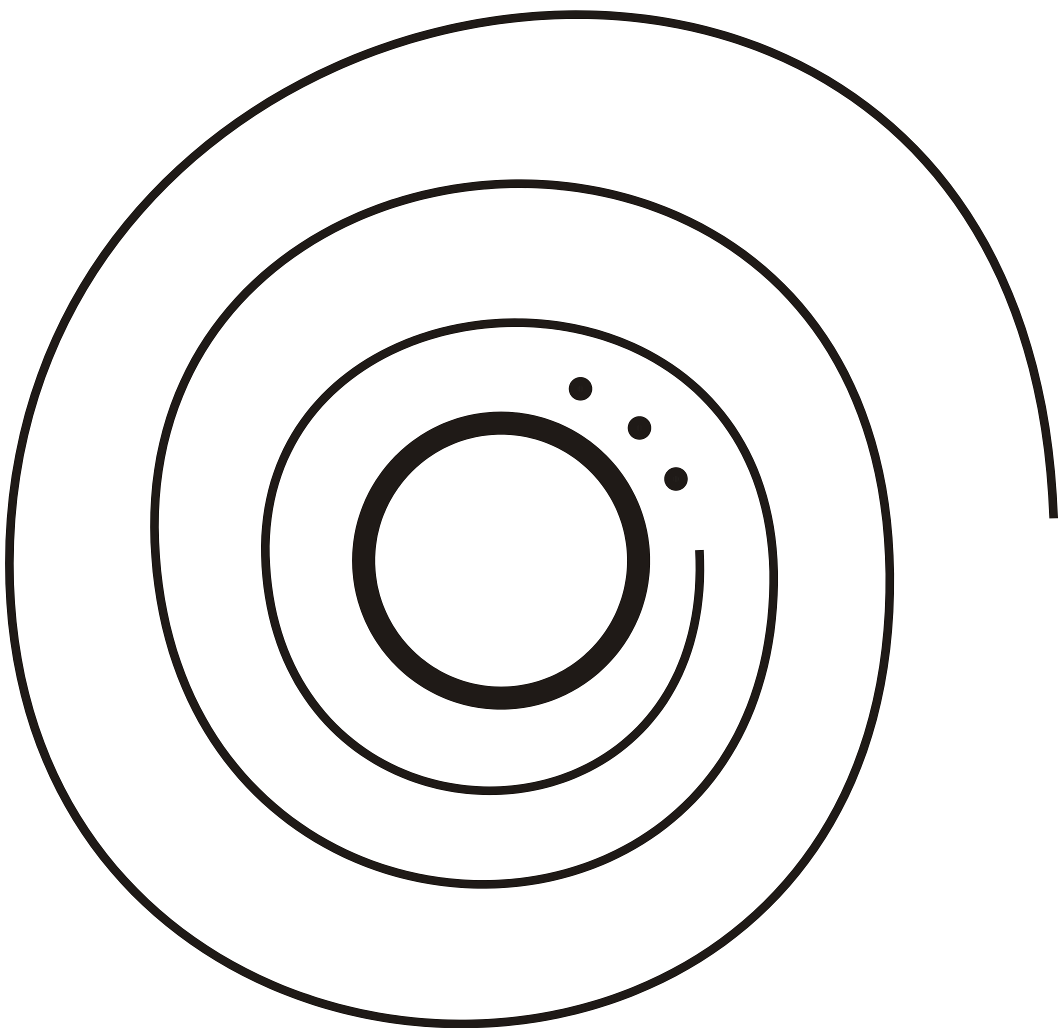
Theorem. For a locally connected continuum X, the following are equivalent:

(a) X is unicoherent,

(b) each mapping $f : X \to S^1$ is homotopic to a constant,

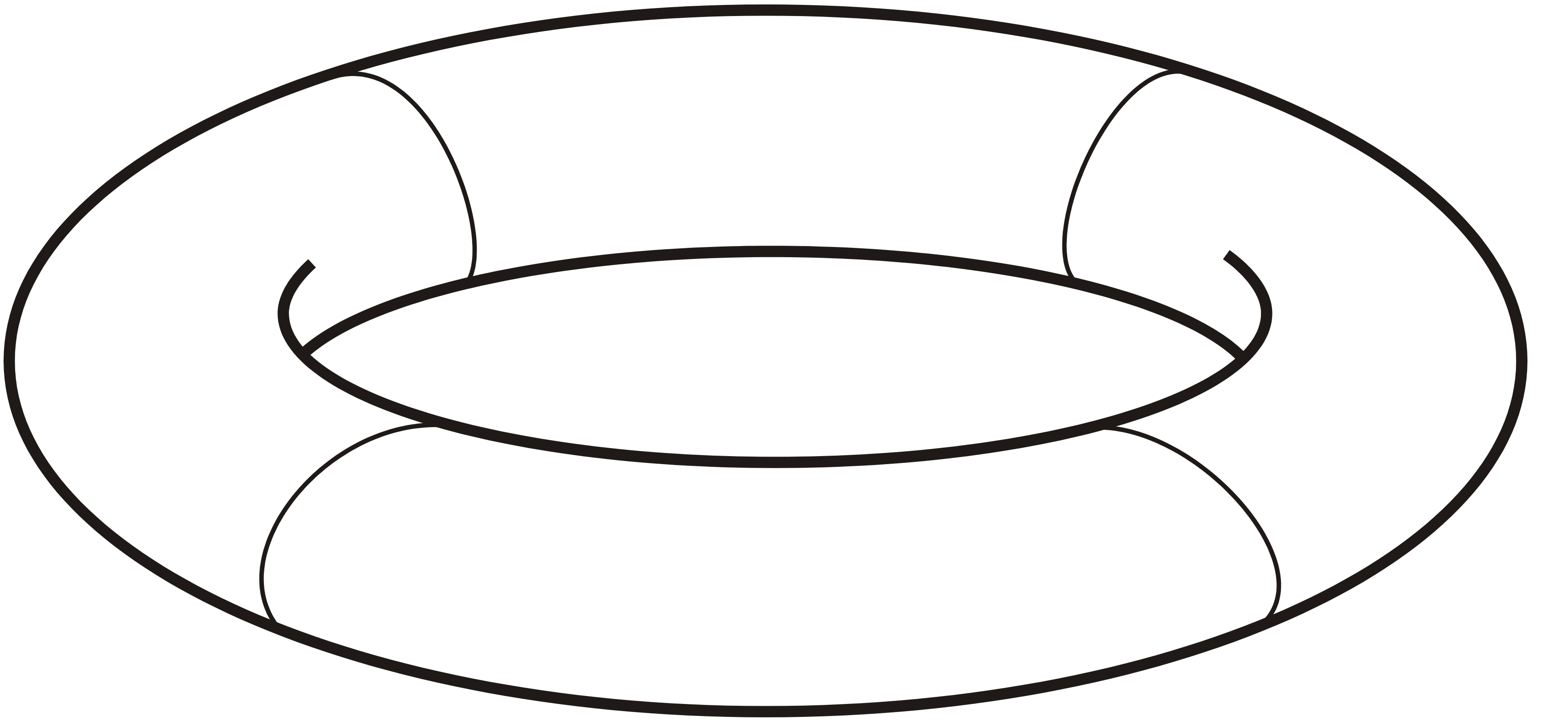
(c) for each mapping $f : X \to S^1$ there exists a mapping $h : X \to \mathbf{R}$ such that $f = e \circ h$.

Z



Theorem. If X and Y are locally connected continua, then $r(X \times Y) = \max\{r(X), r(Y)\}$.

Example (A. Garcia-Maynez and A. Illanes, 1989)**.** $Z \times Z$ is not unicoherent.



Questions (A. Garcia-Maynez and A. Illanes, 1989)**:**

(a) Let $n > 1$, does there exists a unicoherent continuum X such that $r(X \times X) = n$?

(b) Is the product of a unicoherent continuum and a locally connected unicoherent continuum unicoherent?

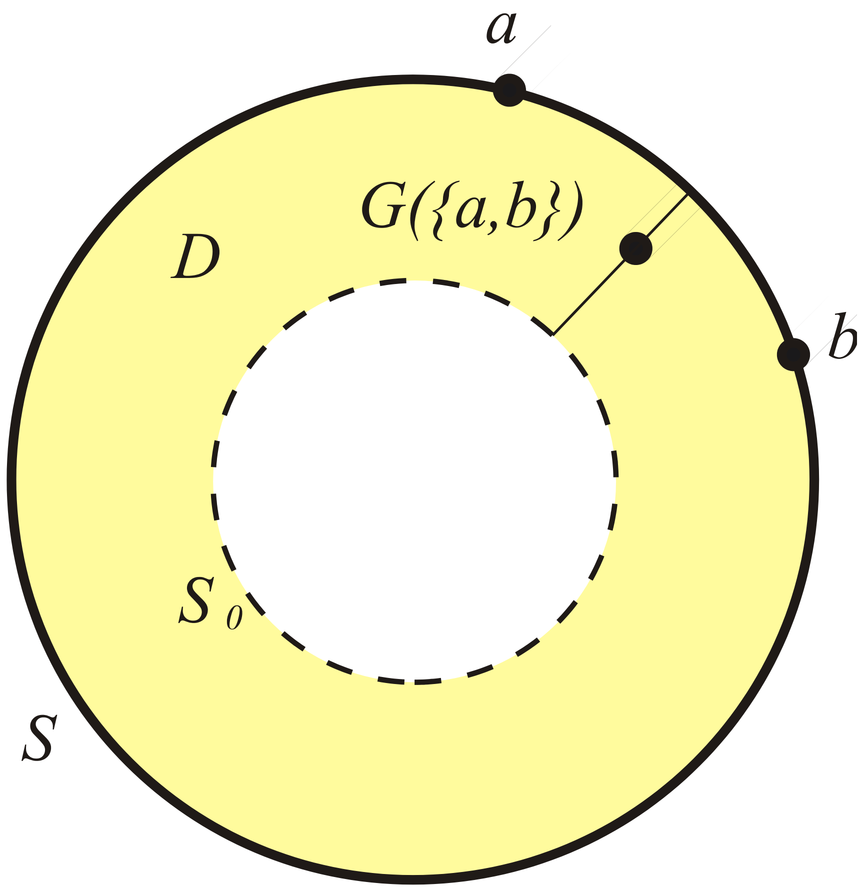
For a continuum X, the **n^{th}-symmetric product** of X is defined as

$X(n) = \{ A \subset X$: A is nonempty and A has at most n points $\}$.

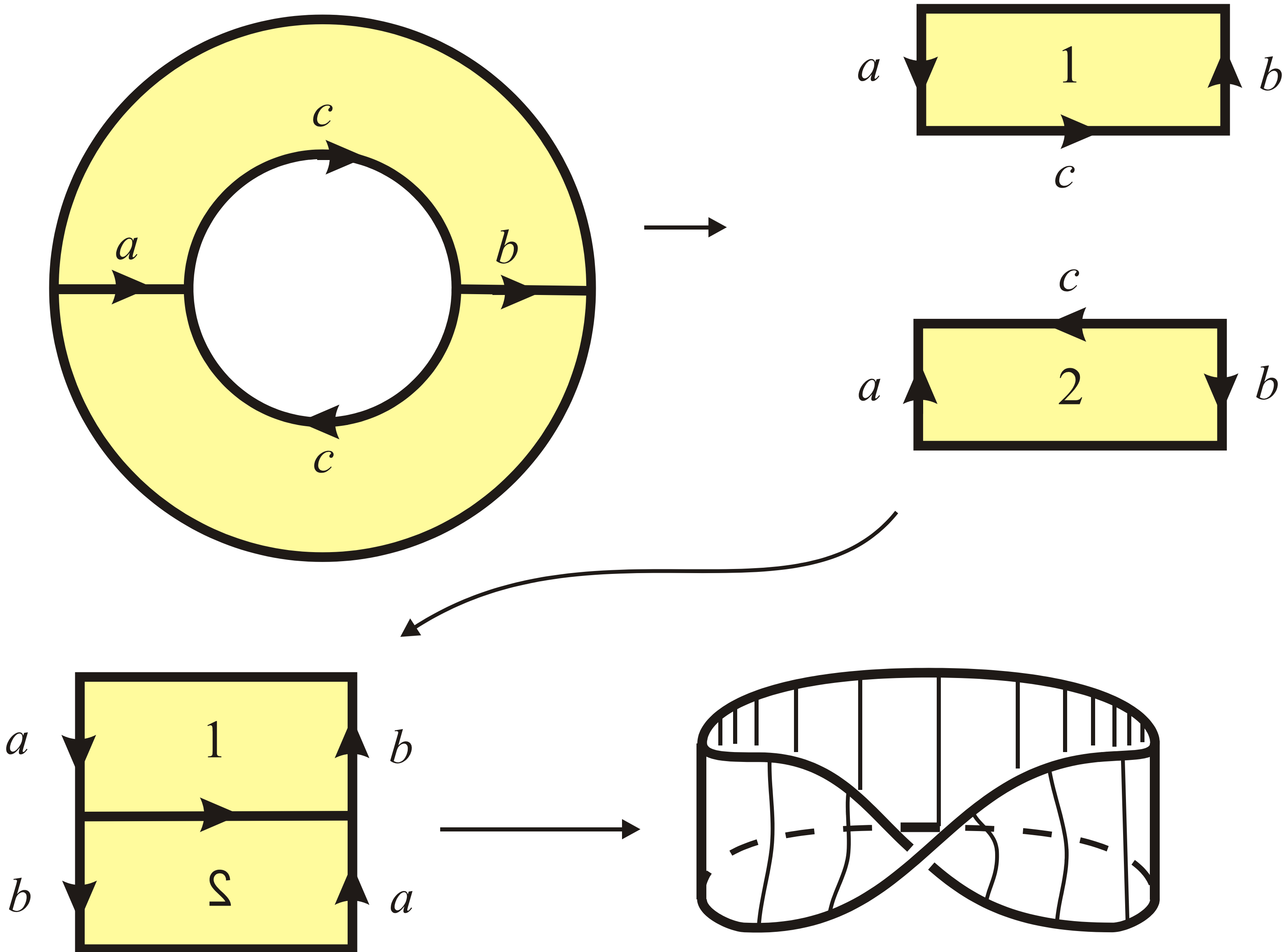
Question (K. Borsuk and S. Ulam, 1931)**.** Is $X(n)$ unicoherent for each locally connected unicoherent continuum X?

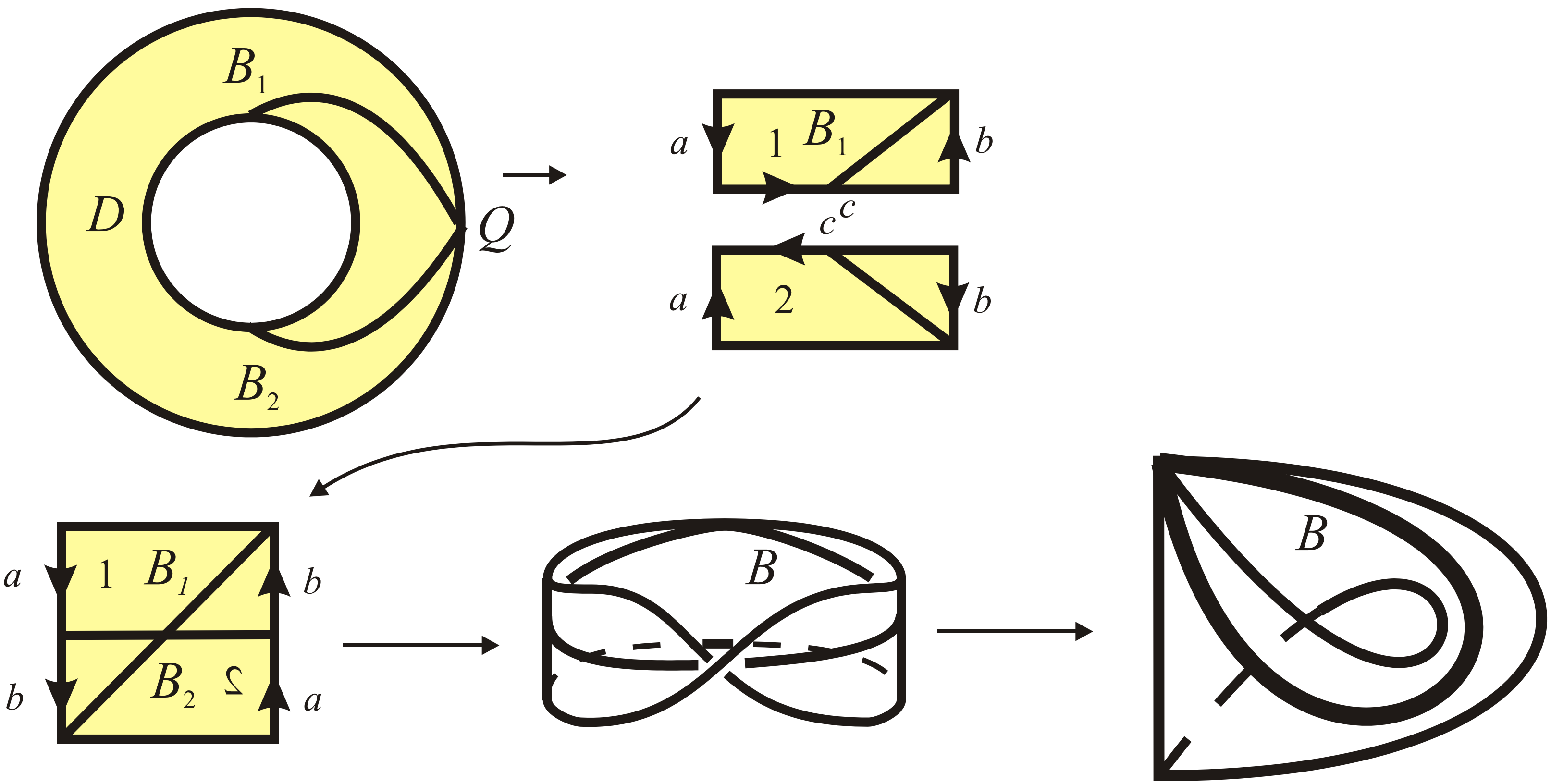
Theorem (T. Ganea, 1954)**.** If X is a locally connected unicoherent continuum, then $X(n)$ is unicoherent for each n.

$F_2(S^1)$



$F_2(S^1)$





B1
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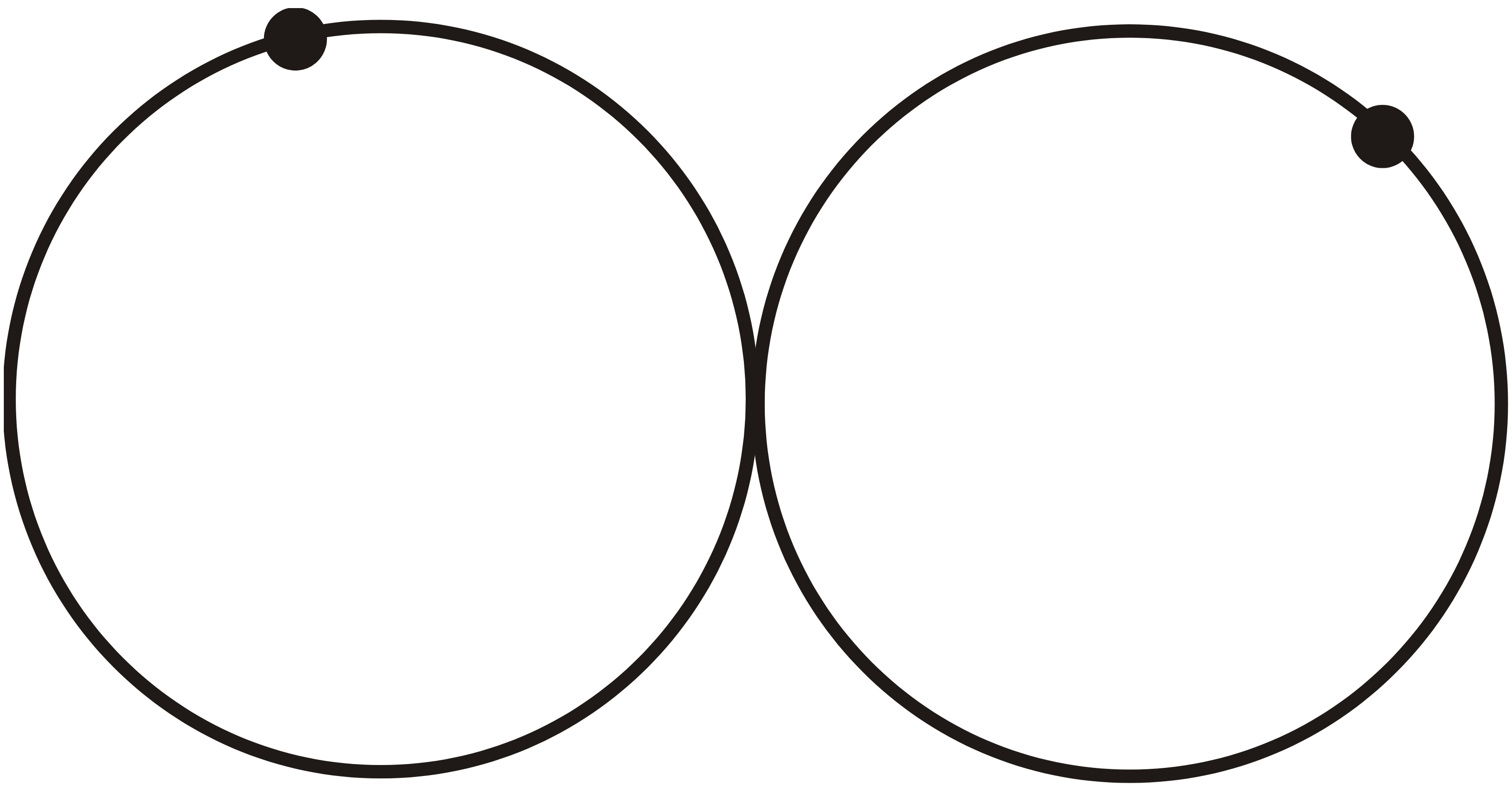
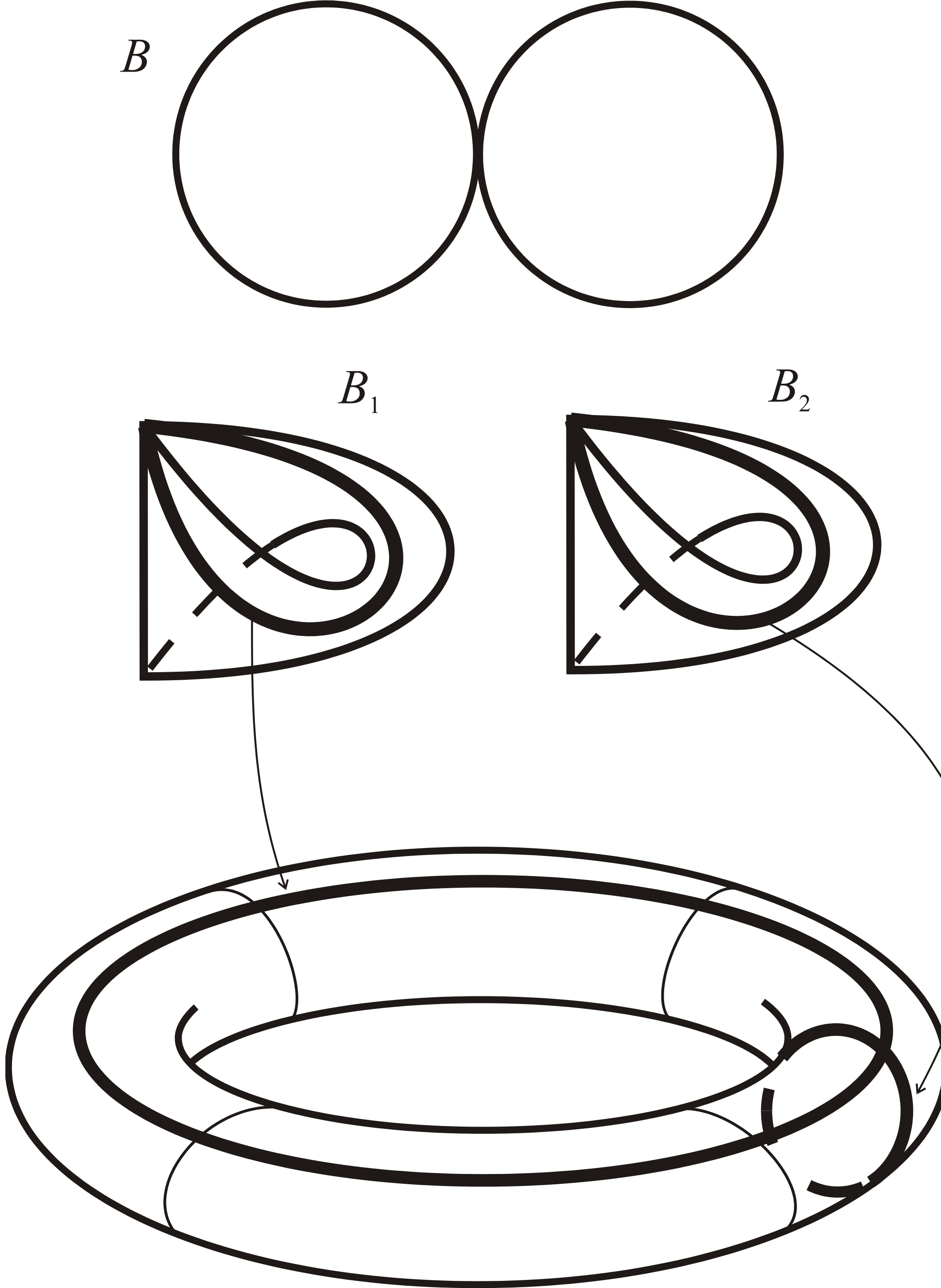
B

Figure eight continuum





B
B_1
B_2

Theorem (A. Illanes, 1985)**.** Let X be a locally connected continuum. Then
(a) if X is unicoherent, then X(2) is unicoherent,
(b) if X is not unicoherent, then r(X(2)) = 1,
(c) X(n) is unicoherent for each n > 2.

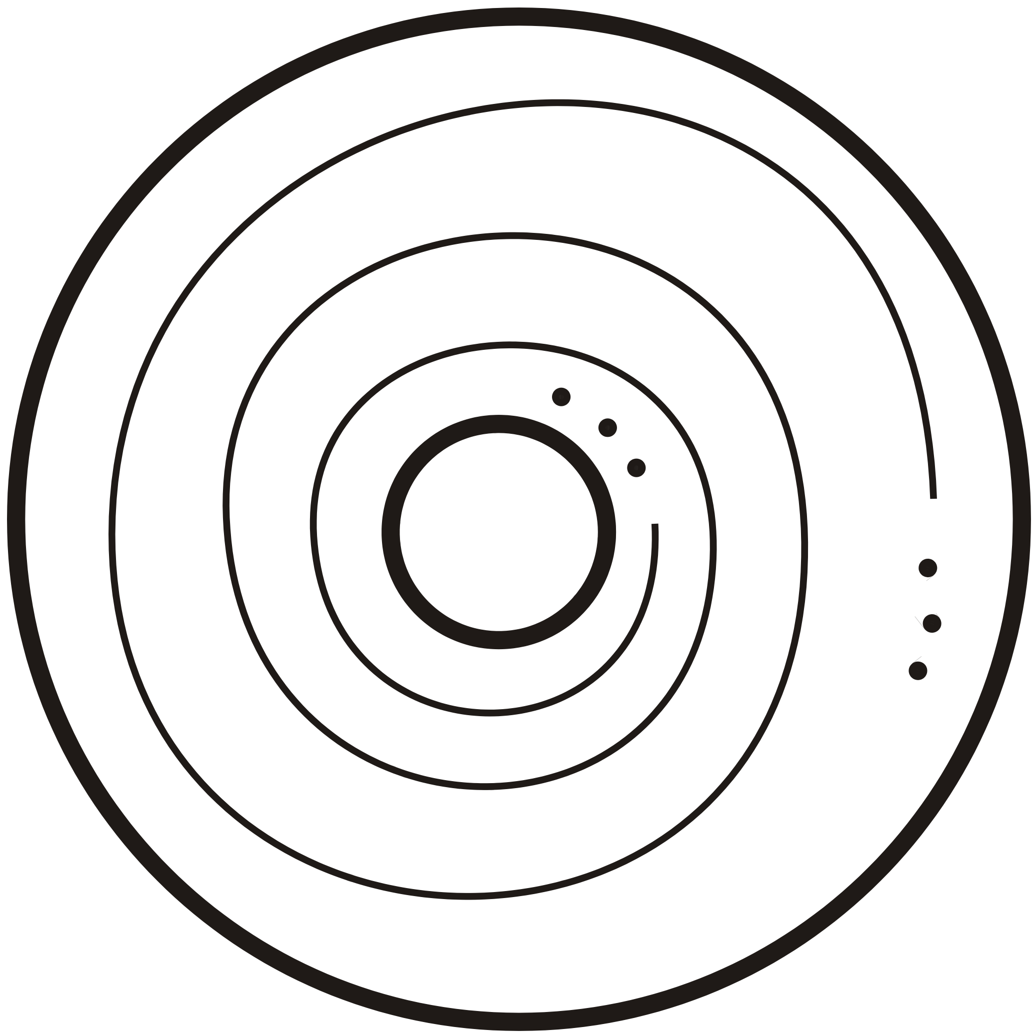
Remark (S. Macias, 1999)**.** For each continuum X:
(a) r(X(2)) $\leq$ 1,
(b) X(n) is unicoherent for each n > 2.

EXAMPLE (K. Borsuk, 1948).
$S^1(3)$ is homeomorphic to $S^1 \times S^2$.

EXAMPLE (Bott, 1951).
$S^1(3)$ is homeomorphic to S^3.

Example (E. Castaňeda, 1998)**.** There exists a unicoherent continuum W such that W(2) is not unicoherent.

W



If $1 \leq m < n$ and X is a continuum, $X(m,n) = X(n)/X(m)$.

Remark. For each $n > 2$ and for every $1 \leq m < n$, $X(m,n)$ is unicoherent.

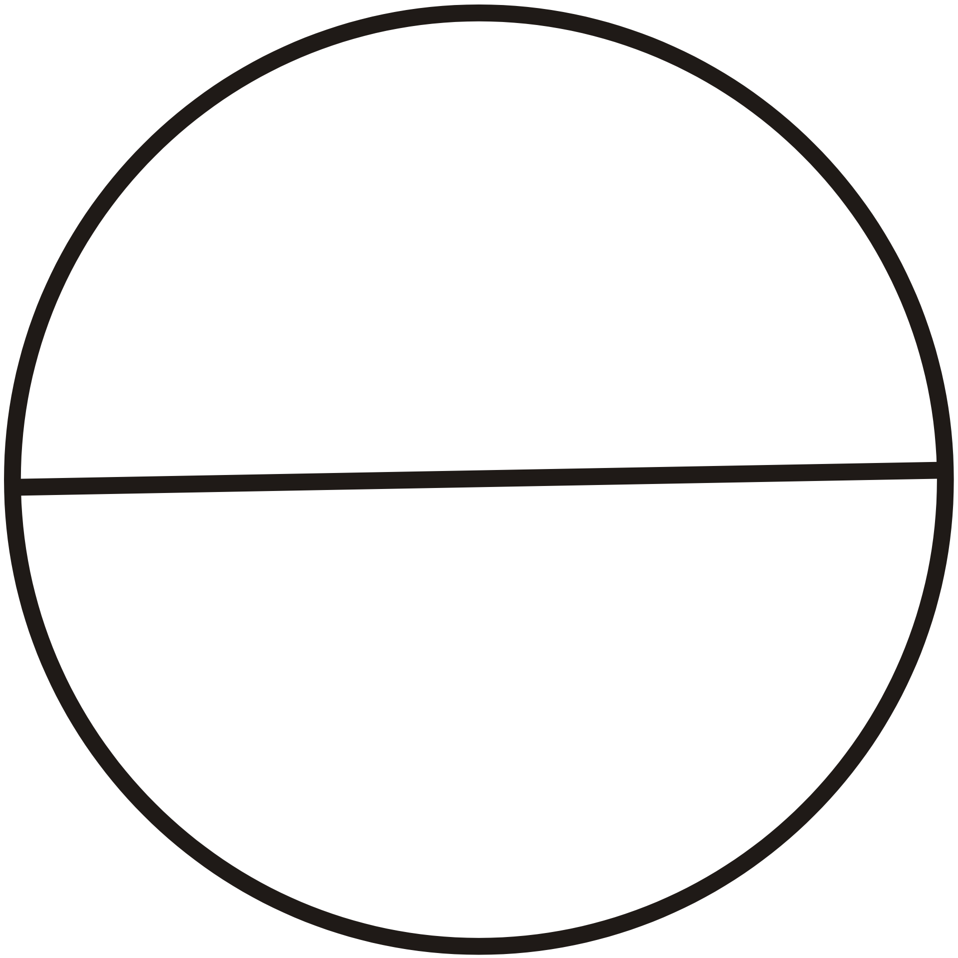
What about $X(1,2)$?

Claim (F. Barragán, 2010)**.** For the Castaneda's continuum W, W(1,2) is not unicoherent.

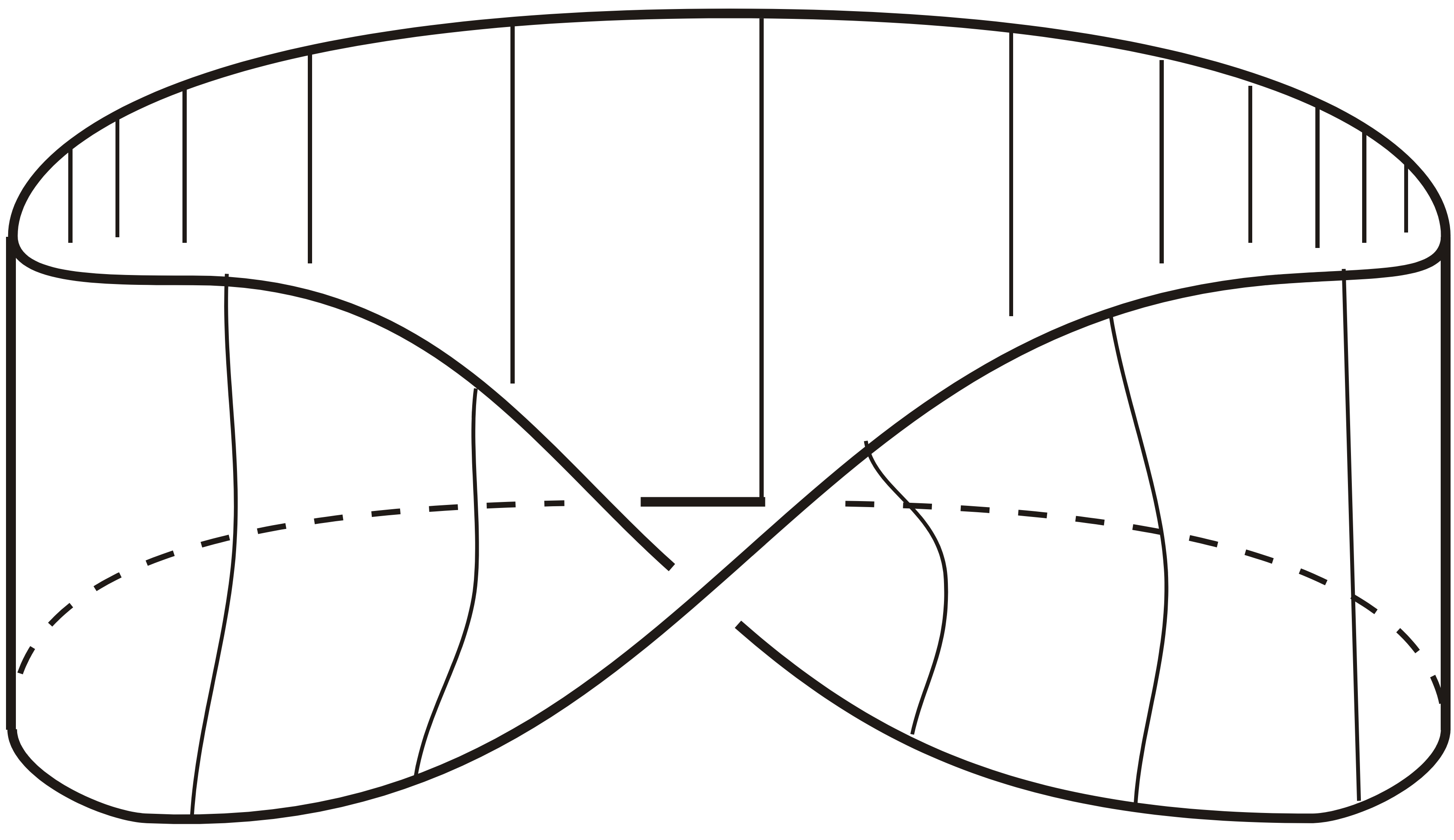
Example (E. Castaneda and J. Martinez, 2013)**.** W(1,2) is unicoherent.

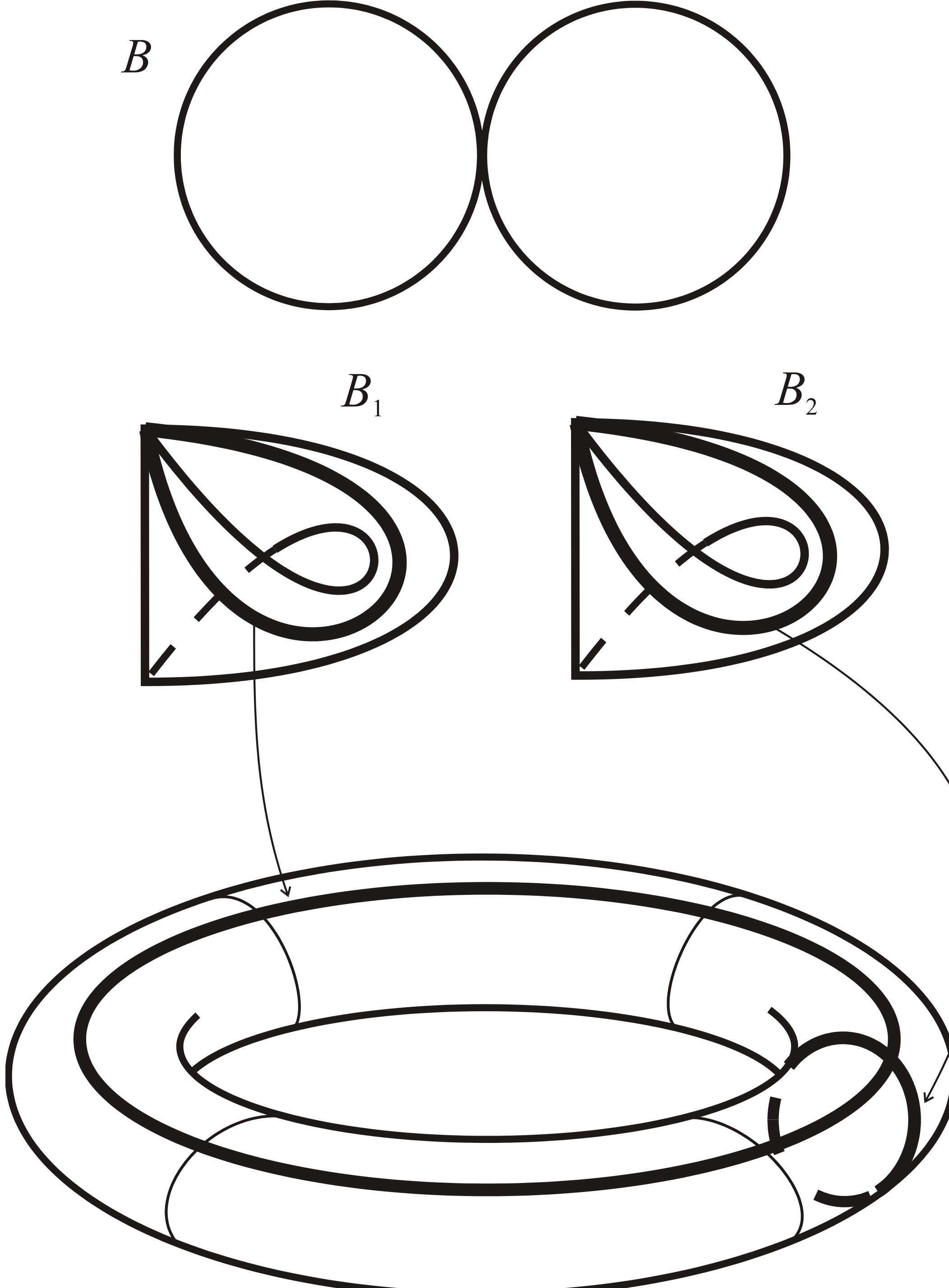
Claim (E. Castaneda and J. Sanchez, 2013)**.**

For the Θ-continuum, Θ(1,2) is not unicoherent.



$S^1(2)$





B
B_1
B_2

Theorem (Castaneda and Illanes, 2013)**.** For every continuum X, X(1,2) is unicoherent.

Lemma. If X is a locally connected continuum and $f : X(2) \to S^1$ is such that $f|X(1) : X(1) \to S^1$ is homotopic to a constant, then f is homotopic to a constant.

Thank you