Topological Entropy of Compact Subsystems of Transitive Real Line Maps

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joint work with:
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North Bay, July 23, 2013

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Theorem

Let f be a transitive map of a real interval J. Then, exactly one of the following statements holds:

1. f^2 is transitive,
2. there exist intervals $K, L \subset J$, with $K \cap L = \{c\}$ and $K \cup L = J$, such that c is the unique fixed point for f, $f(K) = L$ and $f(L) = K$.

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Corollary

If f is a transitive map of a half-open interval, then f is bitransitive.

Horseshoes

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An s-horseshoe for f is a compact interval $J \subset L$, and a collection $\mathcal{C} = \{A_1, \ldots, A_s\}$ of $s \geq 2$ nonempty compact intervals of J fulfilling the following two conditions:

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Tightness vs Looseness

A horseshoe $(J, \mathcal{C})$ is tight if J is the union of elements of $\mathcal{C}$ and $f(A) = J$ for every $A \in \mathcal{C}$.

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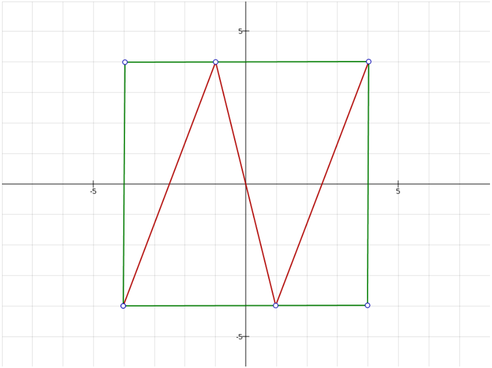

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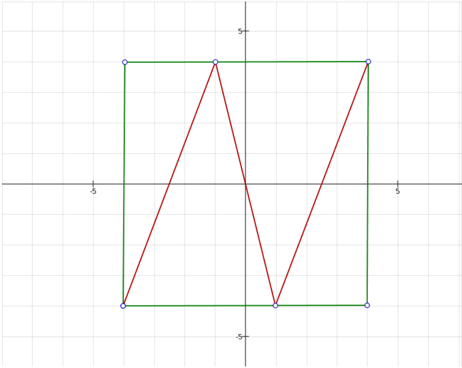

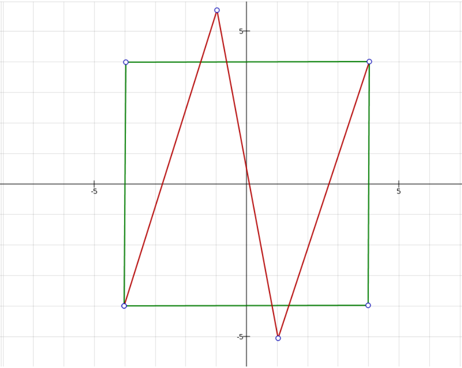

Theorem

If a transitive map f of a real interval L has a loose s-horseshoe then there exists a compact invariant subset K such that $h_{CR}(f) \geq h(f|_K) > \log s$.

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Theorem (DK, MŁ)

If a map g from the half-open interval $[0,\infty)$ to itself is transitive, then g has a loose 3-horseshoe, hence $h_{CR}(g) > \log 3$.

Theorem (DK, MŁ)

If a transitive map of the real line f has at least two fixed points, then f has a loose 2-horseshoe, hence $h_{CR}(f) > \log 2$.

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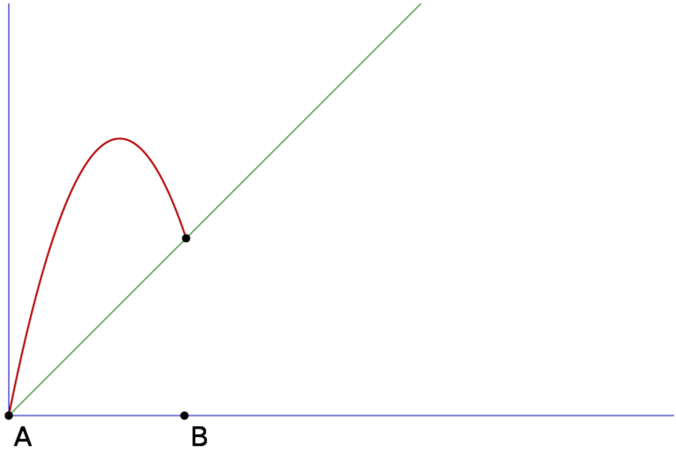

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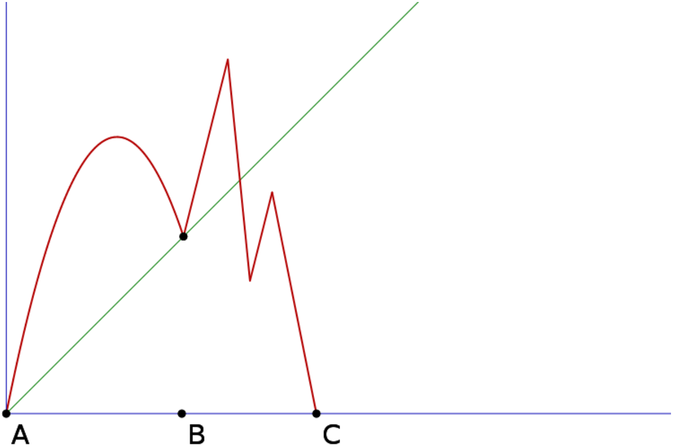

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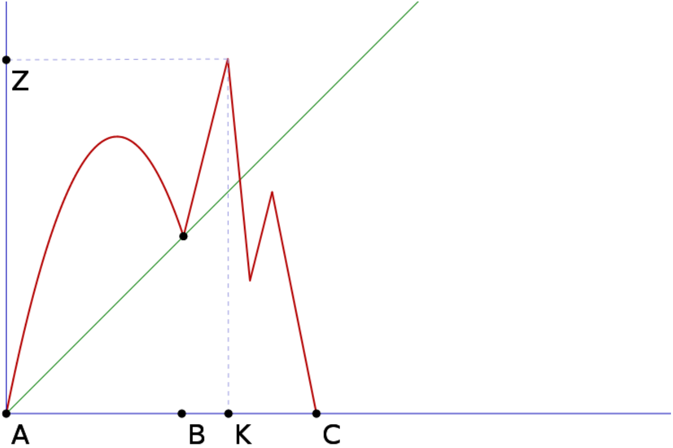

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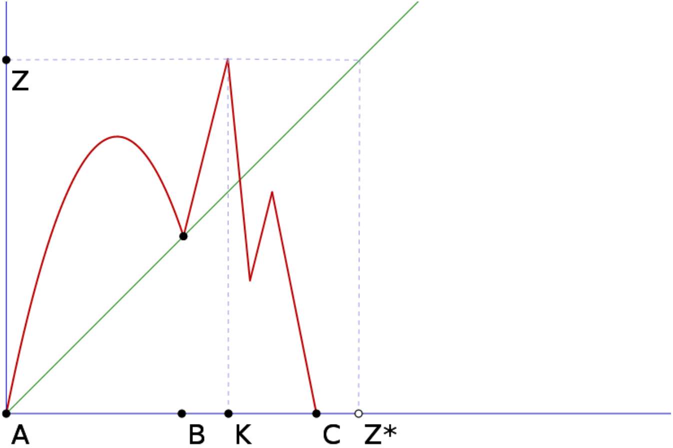

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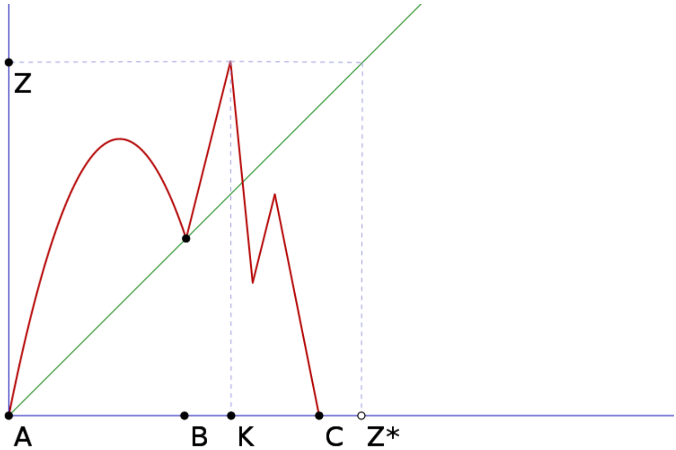

AK, KC form a loose 2-horseshoe for f.

Theorem (DK, MŁ)

If a transitive map f of the real line has a unique fixed point, then f^2 has a loose 3-horseshoe, hence $h_{CR}(f) > \log \sqrt{3}$.

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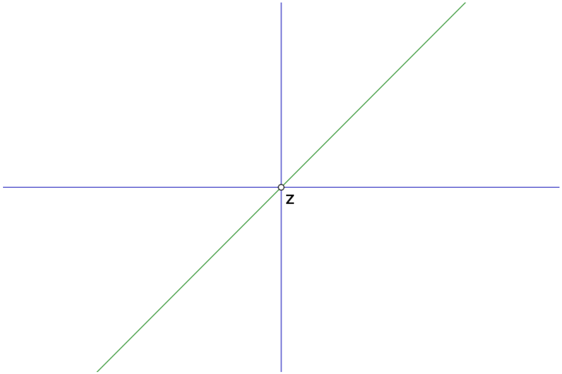

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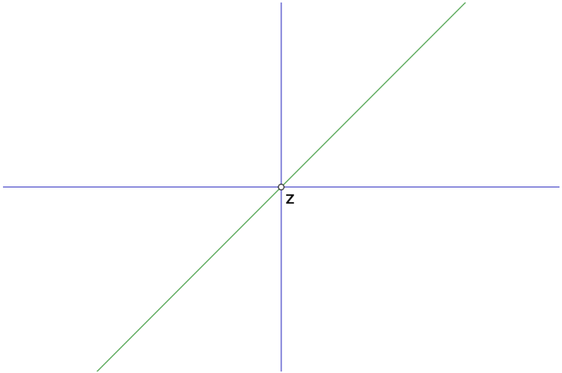

$$Z_- = (-\infty, Z], \quad Z_+ = [Z, +\infty)$$

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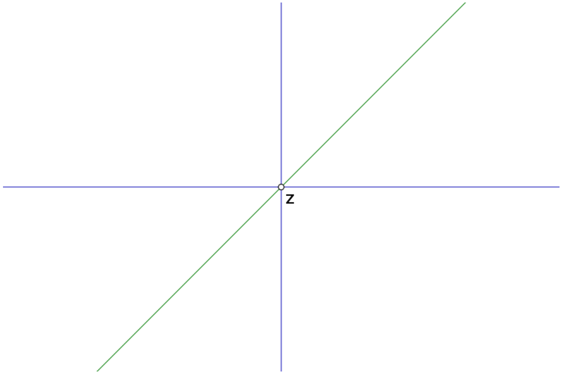

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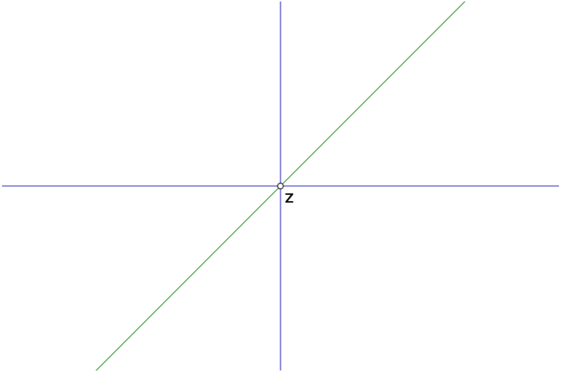

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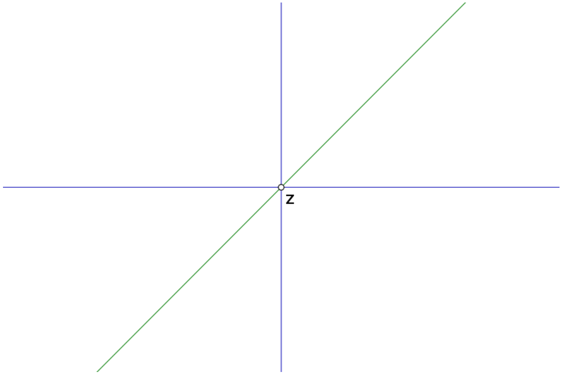

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2. $Z_+ \subsetneq f(Z_-)$ ($\iff$ f is bitransitive).

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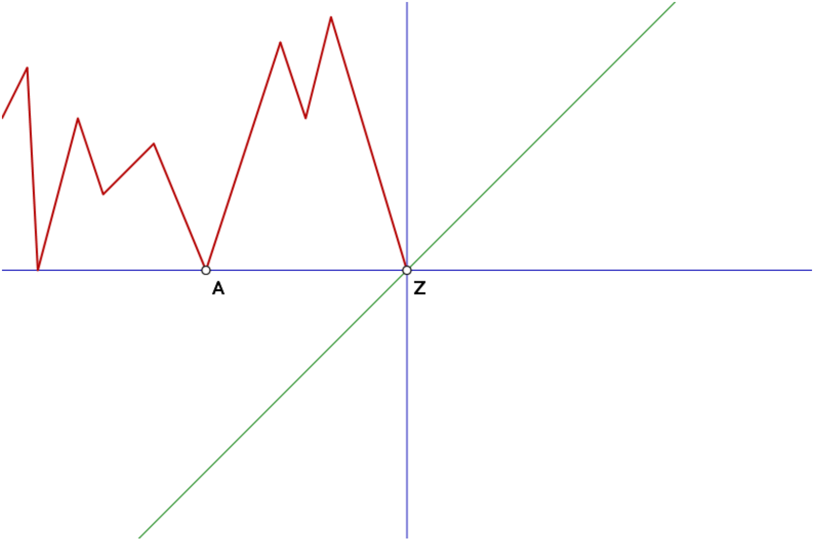

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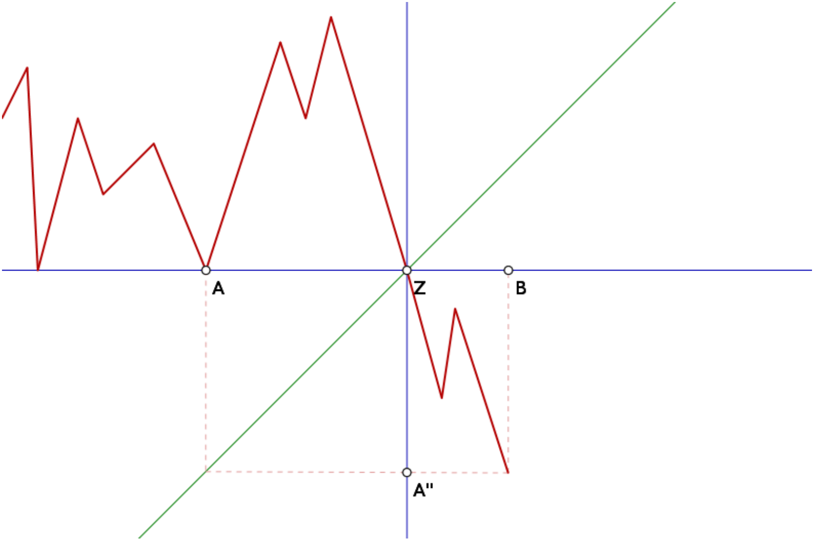

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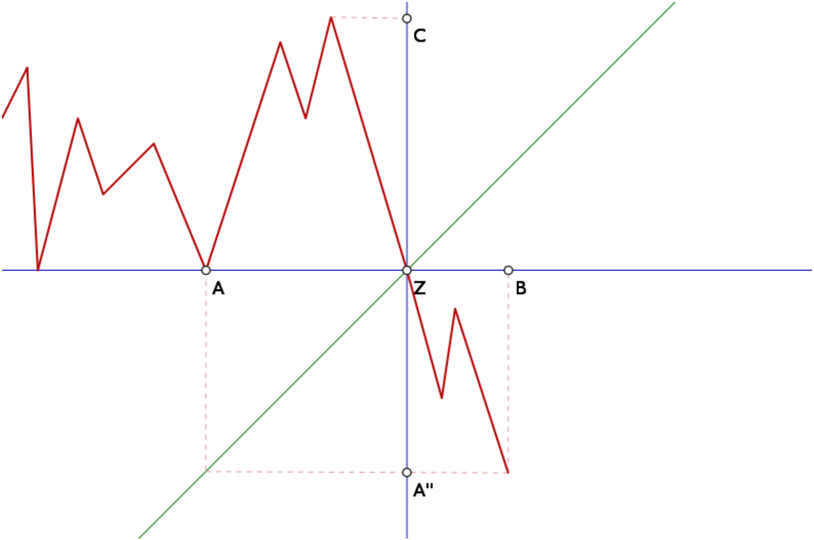

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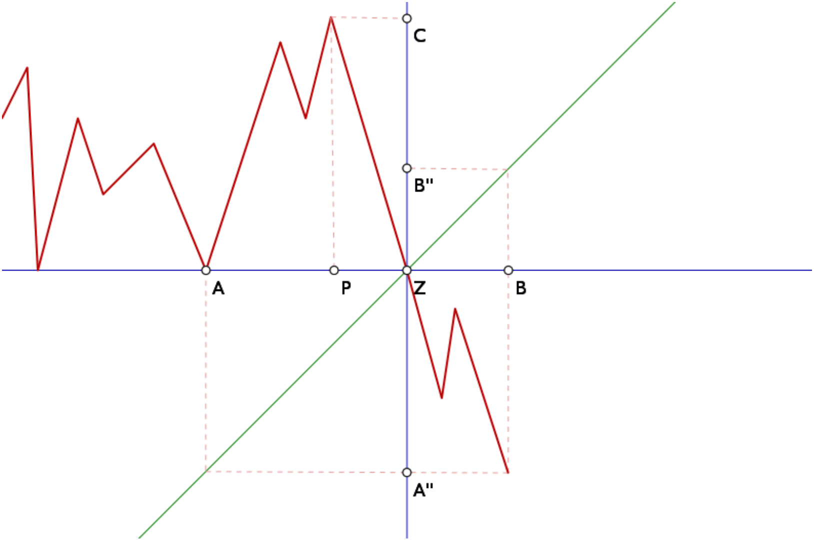

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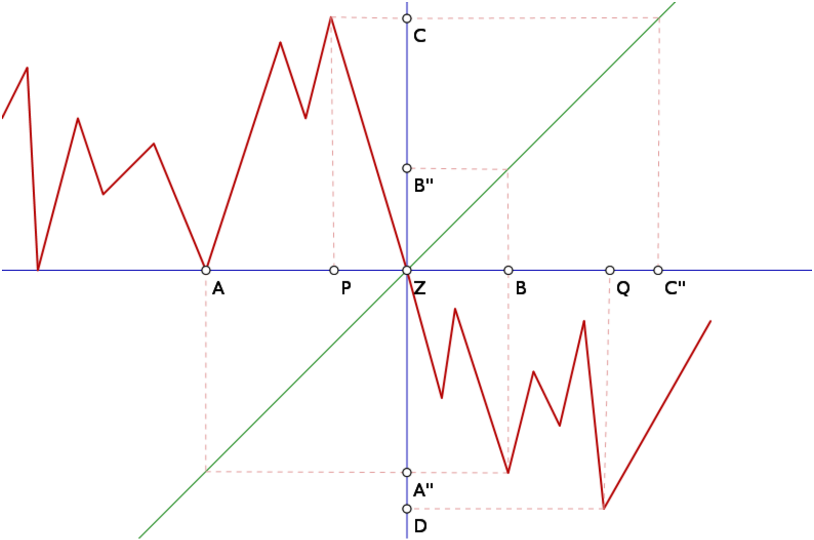

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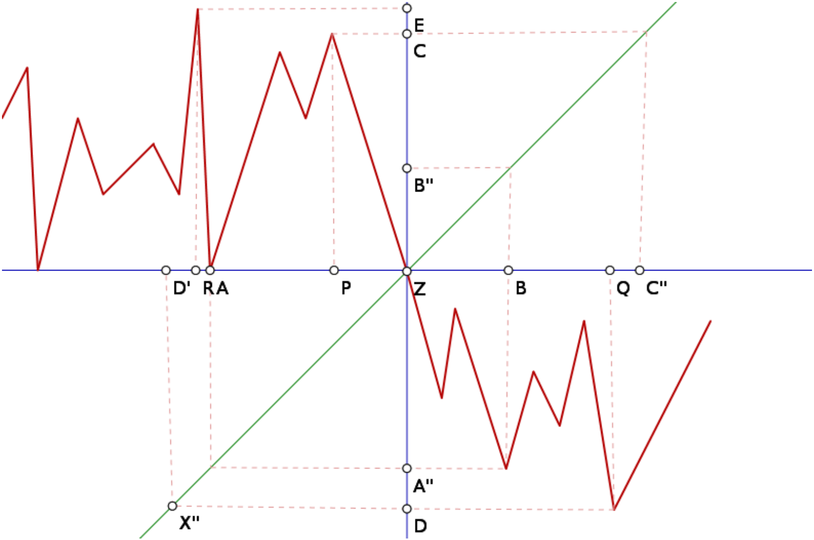

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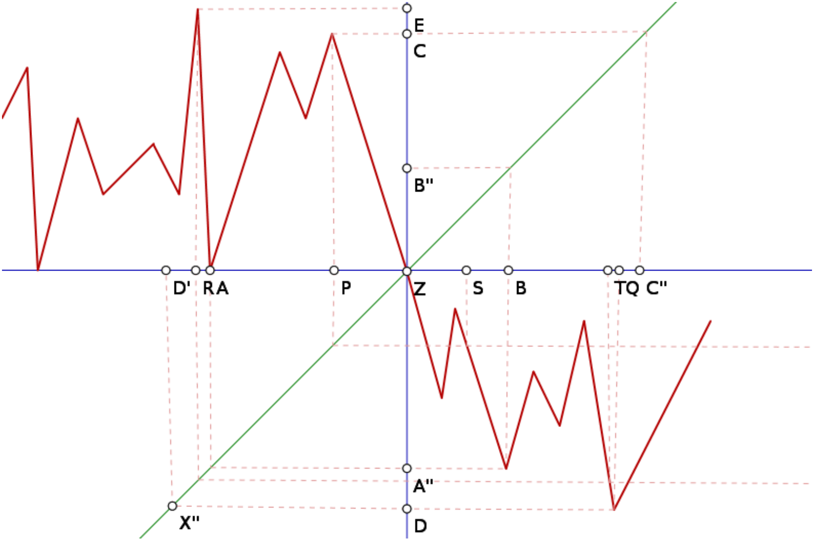
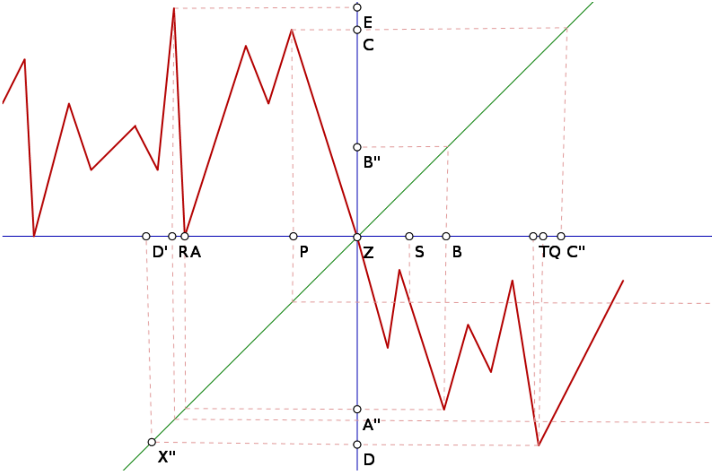
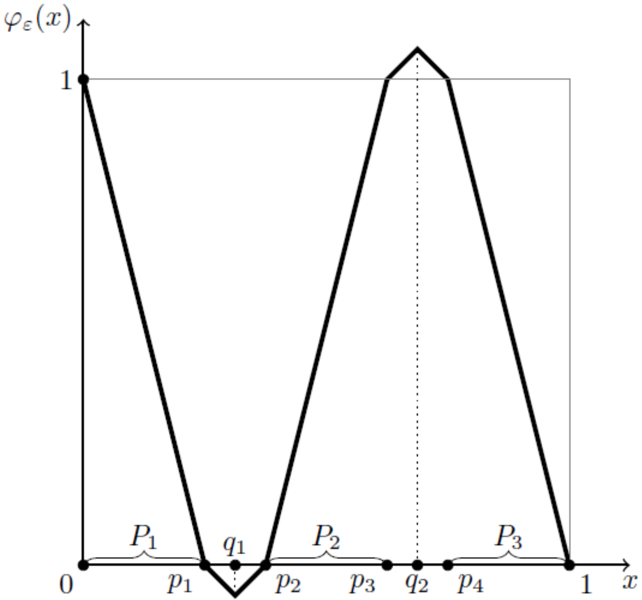
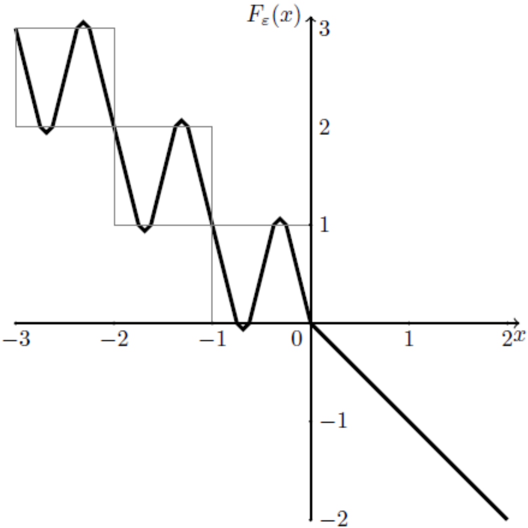


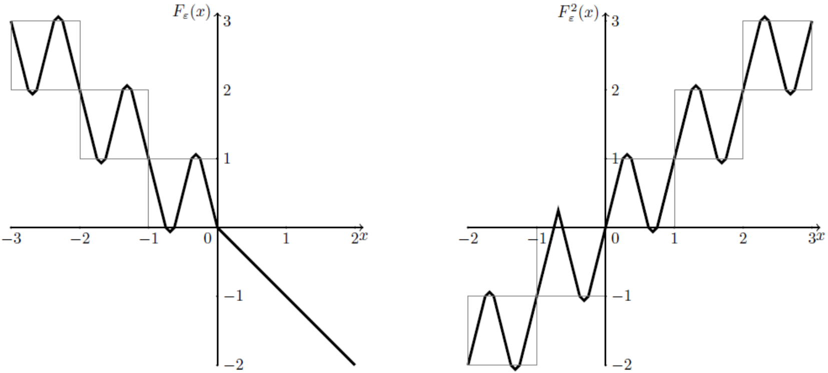


$[Z, S], [S, B], [B, T]$ form a loose $3-$horseshoe for f^2.

$$\inf \{ h_{CR}(f, \mathbb{R}) \quad | \quad f : X \to X \text{ bitransitive } \} = \log \sqrt{3}$$


$\varphi_\varepsilon(x)$
1
P_1
q_1
P_2
P_3
0
p_1
p_2
p_3
q_2
p_4
1
x


$F_\varepsilon(x)$
3
2
1
−3
−2
−1
0
1
2
x
−1
−2



$F_\varepsilon(x)$
3
2
1
−3
−2
−1
0
1
2
x
−1
−2
$F_\varepsilon^2(x)$
3
2
1
−2
−1
0
1
2
3
x
−1
−2

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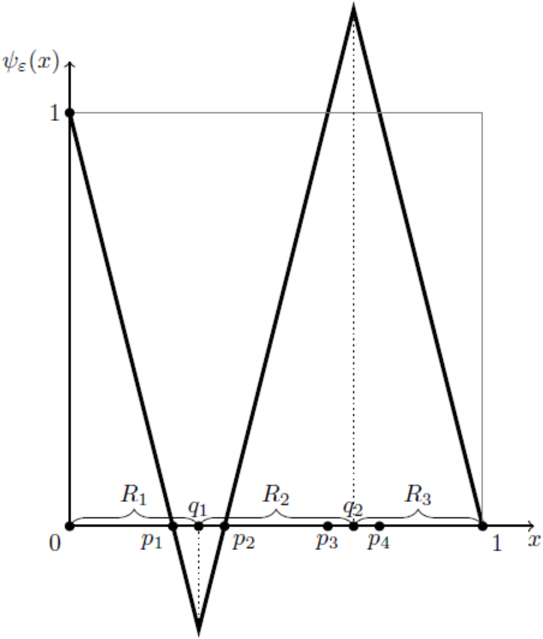
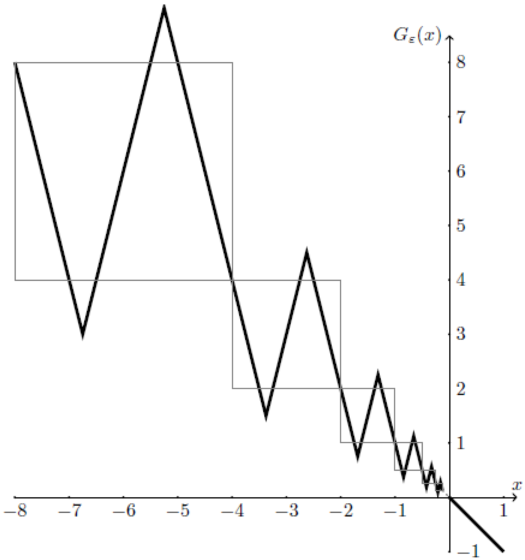
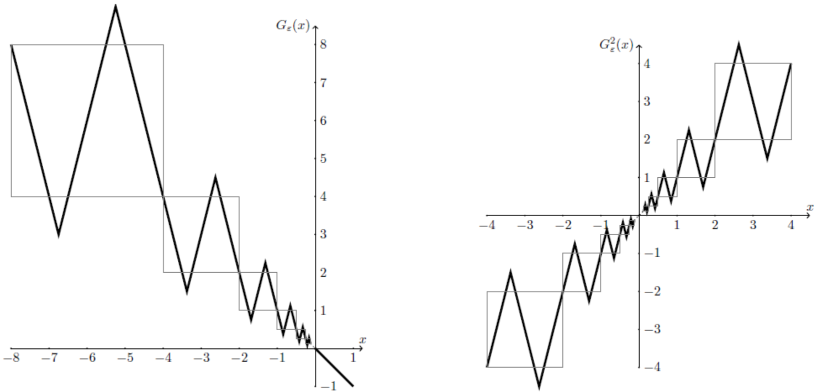

$\psi_\varepsilon(x)$
1
R_1
q_1
R_2
q_2
R_3
0
p_1
p_2
p_3
p_4
1
x


$G_\varepsilon(x)$
8
7
6
5
4
3
2
1
-1
x
-8
-7
-6
-5
-4
-3
-2
-1
1


$G_\varepsilon(x)$
x
$G_\varepsilon^2(x)$
x

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