Topological Variety of Buried Points

Clinton P. Curry, Logan C. Hoehn,
and John C. Mayer

Huntingdon College
Nipissing University
University of Alabama at Birmingham

Summer Topology Conference, Nipissing University, July, 2013

1. Rational Julia Sets
   - Introduction
   - Residual Julia Set = Buried Points

2. Connectedness of Buried Points
   - 0-Dimensional versus Connected
   - Infinitely Many Topological Types

1. Rational Julia Sets
   - Introduction
   - Residual Julia Set = Buried Points

2. Connectedness of Buried Points
   - 0-Dimensional versus Connected
   - Infinitely Many Topological Types

Fatou and Julia

Given: $R : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is a rational function.

- *Fatou set* of R, denoted $F(R)$, is the domain of normality for the family of functions $\{R^i \mid i \in \mathbb{N}\}$.
- A component of the Fatou set is called a *Fatou component*.
- *Julia set* of R, denoted $J(R)$, is the complement of $F(R)$.
- The Julia set is the set with chaotic dynamics; the Fatou set is stable.

Fatou and Julia

Given: $R : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is a rational function.

- *Fatou set* of R, denoted $F(R)$, is the domain of normality for the family of functions $\{R^i \mid i \in \mathbb{N}\}$.
- A component of the Fatou set is called a *Fatou component*.
- *Julia set* of R, denoted $J(R)$, is the complement of $F(R)$.
- The Julia set is the set with chaotic dynamics; the Fatou set is stable.

Fatou and Julia

Given: $R : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is a rational function.

- *Fatou set* of R, denoted $F(R)$, is the domain of normality for the family of functions $\{R^i \mid i \in \mathbb{N}\}$.
- A component of the Fatou set is called a *Fatou component*.
- *Julia set* of R, denoted $J(R)$, is the complement of $F(R)$.
- The Julia set is the set with chaotic dynamics; the Fatou set is stable.

Fatou and Julia

Given: $R : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is a rational function.

- *Fatou set* of R, denoted $F(R)$, is the domain of normality for the family of functions $\{R^i \mid i \in \mathbb{N}\}$.
- A component of the Fatou set is called a *Fatou component*.
- *Julia set* of R, denoted $J(R)$, is the complement of $F(R)$.
- The Julia set is the set with chaotic dynamics; the Fatou set is stable.

Julia and Fatou

- Degree of $R \geq 2 \implies$ Julia set $J(R)$ is a non-empty, compact, perfect subset of $\mathbb{C}_\infty$.
- $J(R)$ is nowhere dense in $\mathbb{C}_\infty$ or equal to $\mathbb{C}_\infty$.
- The Julia set and Fatou set are each *fully invariant* under R, meaning that $R^{-1}(J(R)) = J(R)$ and $R^{-1}(F(R)) = F(R)$.

Julia and Fatou

- Degree of $R \geq 2 \implies$ Julia set $J(R)$ is a non-empty, compact, perfect subset of $\mathbb{C}_\infty$.
- $J(R)$ is nowhere dense in $\mathbb{C}_\infty$ or equal to $\mathbb{C}_\infty$.
- The Julia set and Fatou set are each *fully invariant* under R, meaning that $R^{-1}(J(R)) = J(R)$ and $R^{-1}(F(R)) = F(R)$.

Julia and Fatou

- Degree of $R \geq 2 \implies$ Julia set $J(R)$ is a non-empty, compact, perfect subset of $\mathbb{C}_\infty$.
- $J(R)$ is nowhere dense in $\mathbb{C}_\infty$ or equal to $\mathbb{C}_\infty$.
- The Julia set and Fatou set are each *fully invariant* under R, meaning that $R^{-1}(J(R)) = J(R)$ and $R^{-1}(F(R)) = F(R)$.

Continuum Theory

- We are interested in the case when the Julia set is **not** all of $\mathbb{C}_\infty$.
- We are interested in the case when the Julia set is **connected**.
- Under these assumptions, the Julia set is a **one-dimensional continuum**.

Continuum Theory

- We are interested in the case when the Julia set is **not** all of $\mathbb{C}_\infty$.
- We are interested in the case when the Julia set is **connected**.
- Under these assumptions, the Julia set is a **one-dimensional continuum**.

Continuum Theory

- We are interested in the case when the Julia set is **not** all of $\mathbb{C}_\infty$.
- We are interested in the case when the Julia set is **connected**.
- Under these assumptions, the Julia set is a **one-dimensional continuum**.

Residual Julia Set

Definition (Residual Julia Set)

Let $\mathcal{F}$ be the collection of components of the Fatou set $F(R)$. We define the *residual Julia set* as

$$\mathrm{Bur}(J(R)) = J(R) \setminus \bigcup_{F \in \mathcal{F}} \partial F.$$

- The residual Julia set $\mathrm{Bur}(J(R))$ is sometimes called the set of *buried points* of $J(R)$.
- That is, a point of the Julia set is *buried* if it is not in the boundary of any Fatou component.

Residual Julia Set

Definition (Residual Julia Set)

Let $\mathcal{F}$ be the collection of components of the Fatou set $F(R)$. We define the *residual Julia set* as

$$\mathrm{Bur}(J(R)) = J(R) \setminus \bigcup_{F \in \mathcal{F}} \partial F.$$

- The residual Julia set $\mathrm{Bur}(J(R))$ is sometimes called the set of *buried points* of $J(R)$.
- That is, a point of the Julia set is *buried* if it is not in the boundary of any Fatou component.

Residual Julia Set

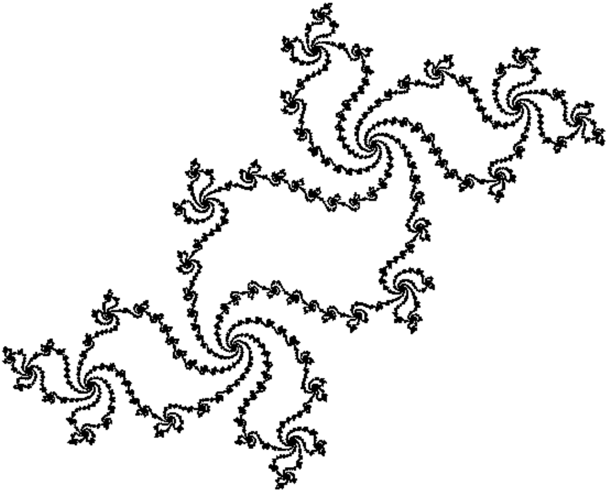
Definition (Residual Julia Set)

Let $\mathcal{F}$ be the collection of components of the Fatou set $F(R)$. We define the *residual Julia set* as

$$\mathrm{Bur}(J(R)) = J(R) \setminus \bigcup_{F \in \mathcal{F}} \partial F.$$

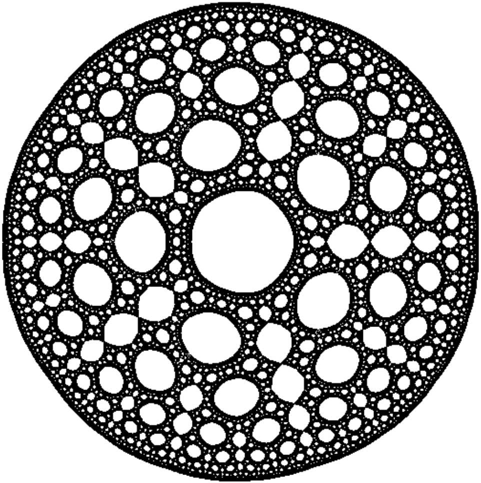
- The residual Julia set $\mathrm{Bur}(J(R))$ is sometimes called the set of *buried points* of $J(R)$.
- That is, a point of the Julia set is *buried* if it is not in the boundary of any Fatou component.

Polynomials have empty residual sets



Michael Becker, http://www.ijon.de/index.html

Some rational functions do not



Michael Becker, http://www.ijon.de/index.html

Residual Julia Set

- The residual Julia set is non-empty iff the boundary of each Fatou component is nowhere dense in $J(R)$.
- Baire Category Theorem $\Longrightarrow$ if not empty, $\mathrm{Bur}(J(R))$ is a dense G_δ subset of the Julia set $J(R)$.
- The residual Julia set and the union of boundaries of Fatou components are each fully invariant subsets of $J(R)$.

Residual Julia Set

- The residual Julia set is non-empty iff the boundary of each Fatou component is nowhere dense in $J(R)$.
- Baire Category Theorem $\Longrightarrow$
  if not empty, $\mathrm{Bur}(J(R))$ is a dense G_δ subset of the Julia set $J(R)$.
- The residual Julia set and the union of boundaries of Fatou components are each fully invariant subsets of $J(R)$.

Residual Julia Set

- The residual Julia set is non-empty iff the boundary of each Fatou component is nowhere dense in $J(R)$.
- Baire Category Theorem $\implies$ if not empty, $\mathrm{Bur}(J(R))$ is a dense G_δ subset of the Julia set $J(R)$.
- The residual Julia set and the union of boundaries of Fatou components are each fully invariant subsets of $J(R)$.

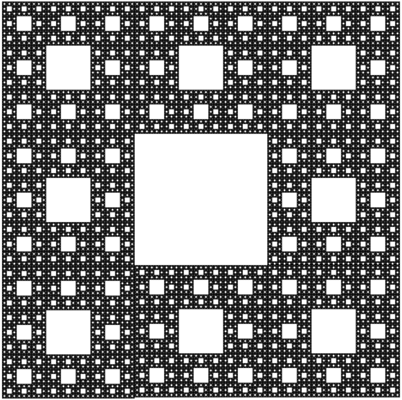
Julia Sets with Buried Points

Functions of the form $z \mapsto z^n + \dfrac{\lambda}{z^d}$ $(n, d \geq 2)$

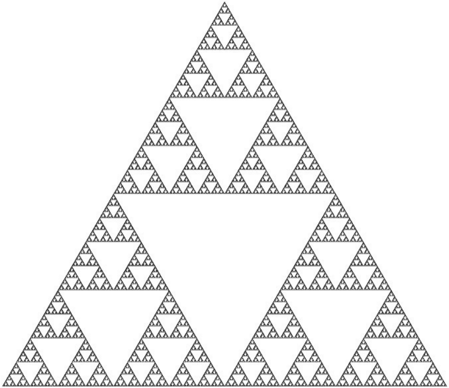
- Some have Julia sets homeomorphic to the Sierpinski carpet [Milnor/Tan:1993; Devaney:2005].
- Some have Julia sets homeomorphic to a generalized Sierpinski gasket [Devaney/Rocha:2007].

"Singular" perturbations of $z \mapsto z^n$. [Devaney]

"Standard" Models for Julia sets with Buried Points



The Sierpinski carpet



The Sierpsinski gasket

σ-Fatou Domain

Definition

A collection $\mathcal{F}$ of Fatou domains will be called *σ-Fatou* if it is maximal with respect to the property that for every $F_1, F_2 \in \mathcal{F}$ there exist finitely many Fatou domains $U_0, \ldots, U_n$ such that $U_0 = F_1$, $U_n = F_2$, and $\overline{U_{i-1}} \cap \overline{U_i} \neq \emptyset$ for each $1 \leq i \leq n$.

- Observe that the σ-Fatou collections form a partition of the set of all Fatou domains.
- Compare carpet and gasket.

Theorem

Let J be a continuum in the sphere S. Suppose

1. *for every $\varepsilon > 0$, there are only finitely many σ-Fatou collections $\mathcal{F}$ such that $\bigcup \mathcal{F}$ has diameter $\geq \varepsilon$;*
2. *for any σ-Fatou collection $\mathcal{F}$, its closure $\overline{\bigcup \mathcal{F}}$ does not separate S;*
3. *the closures of σ-Fatou collections are pairwise disjoint; and*
4. *the closures of σ-Fatou collections are dense in J.*

Then the buried point set of J contains a dense G_δ subset homeomorphic to the buried point set of the Sierpinski carpet.

The $3,3$ Family – degree 6

For definiteness we consider the family

$$z \mapsto z^3 + \frac{\lambda}{z^3}$$

- z is the dynamical variable – Julia sets live in *dynamical space.*
- λ is the parameter – λ lives in *parameter space.*

The $3, 3$ Family – degree 6

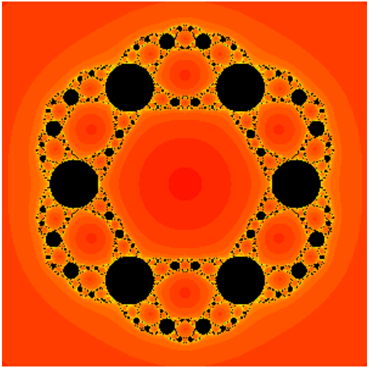
For definiteness we consider the family

$$z \mapsto z^3 + \frac{\lambda}{z^3}$$

- z is the dynamical variable – Julia sets live in *dynamical space.*
- λ is the parameter – λ lives in *parameter space.*

Zero-Dimensional Buried Point Set

"Checkerboard" Julia set

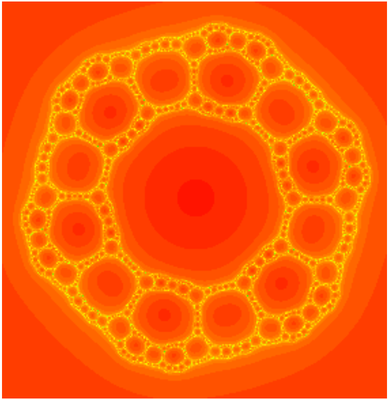


Gasket type – 0-dimensional buried point set – homeomorphic to the irrationals.

Pictures by Bob Devaney's programs.

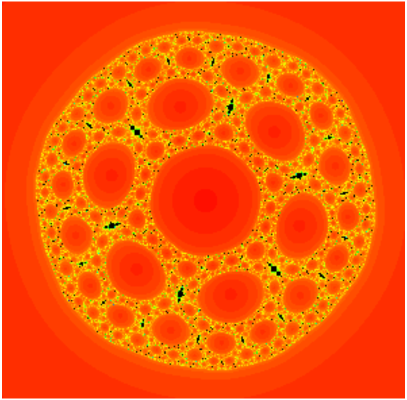
Connected Buried Point Set

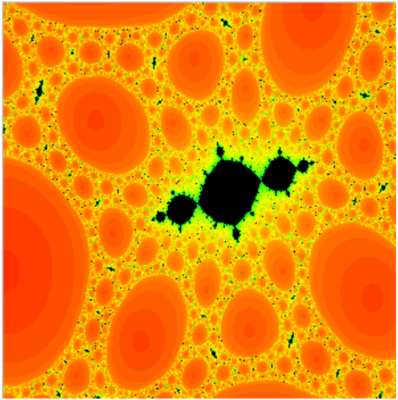
"Very" connected Julia set



Carpet type – buried point set homeomorphic to Sierpinski carpet.

Nontrivial σ-Fatou domains



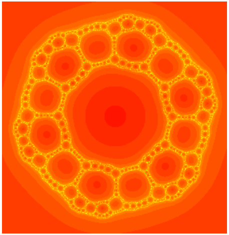


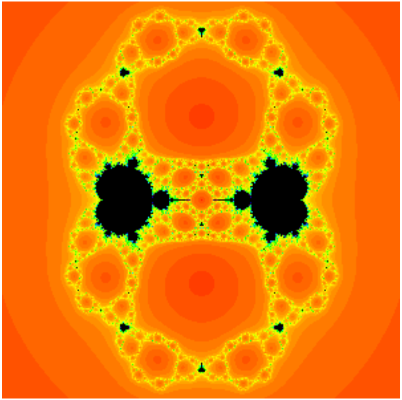
Buried point set contains a dense G_δ subset homeomorphic to Sierpinski carpet.

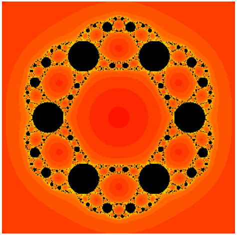
Connectedness of Buried Points

- Is there anything "in between" topologically?

λ Parameter Space





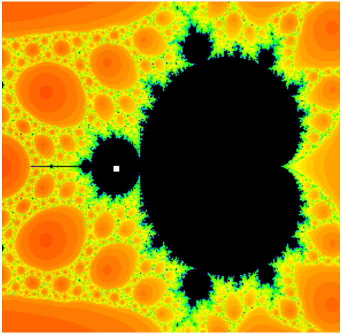


Main cardioid on right.

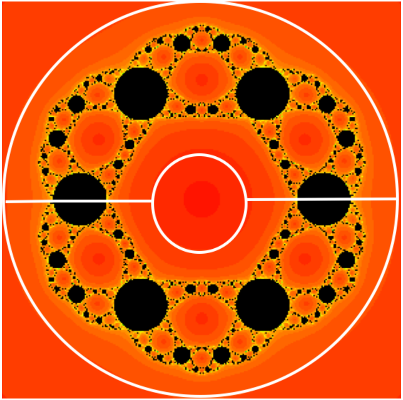
0-Dimensional Buried Point Set

Theorem

Let λ be either in the main cardioid or the result of one satellite bifurcation from the main cardioid. Then the buried point set is zero-dimensional.

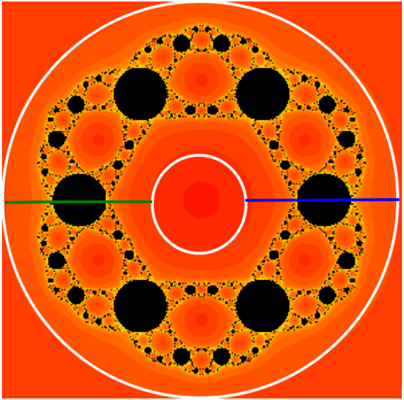


Checkerboard Puzzle Piece Basis



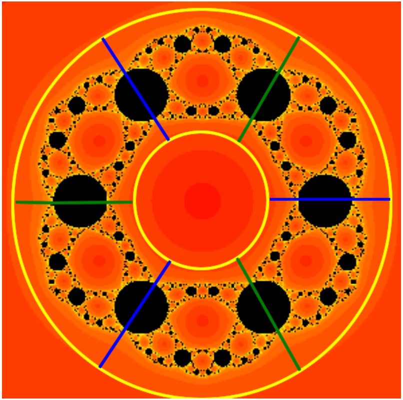
First stage of puzzle piece basis for buried points.

Checkerboard Puzzle Piece Basis



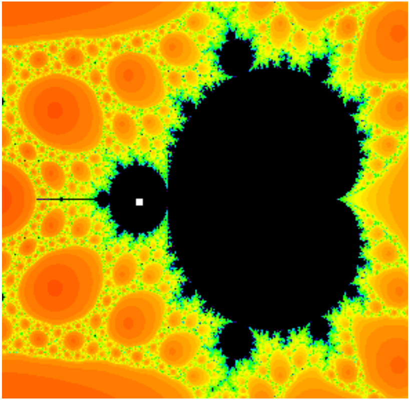
Distinguish ruled arcs.

Checkerboard Puzzle Piece Basis

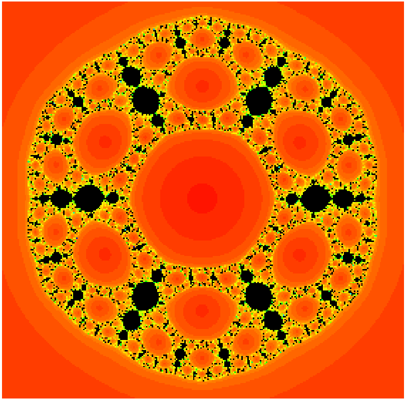


Pullback stage 1 puzzle pieces.

Bifurcations in the Main Cardiod

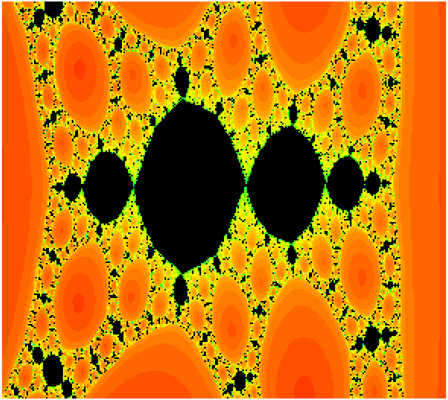


2-bulb off main cardioid,



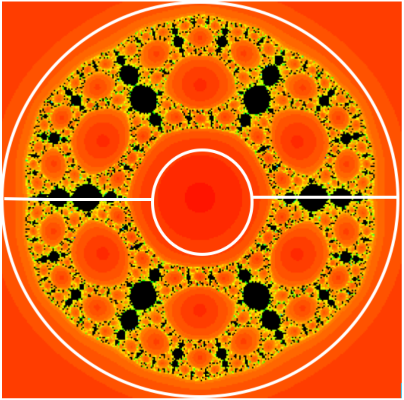
Embedded basillicas,

Bifurcations in the Main Cardiod



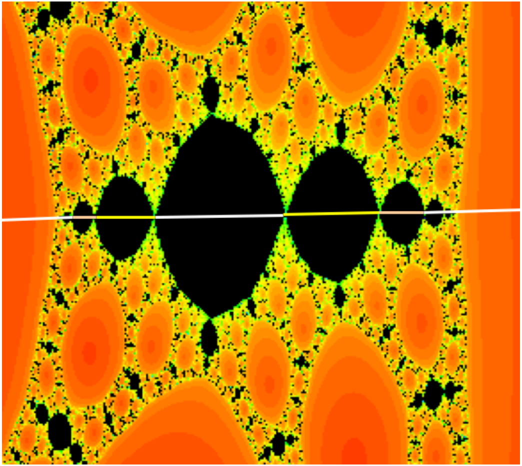
Blow-up of embedded basillica Julia set.

Basillica Puzzle Piece Basis



Ruled arc all the way through the basillica.
Buried point set still 0-dimensional.

Bifurcations in the Main Cardiod

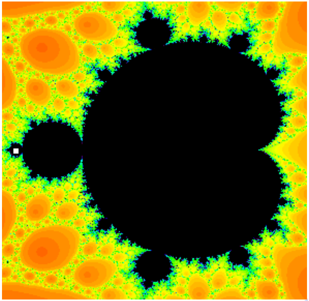


Blow-up of embedded basillica, with ruled arc.

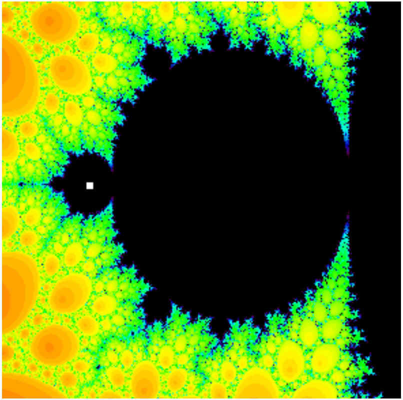
"Suddenly" Connected Buried Points

Theorem

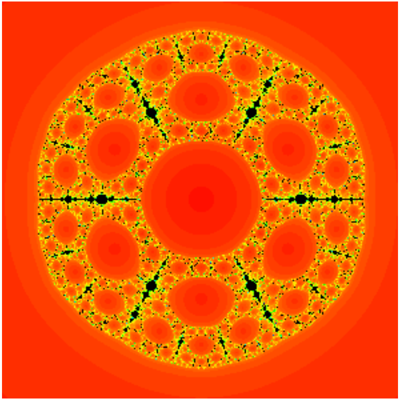
If λ is the result of at least two satellite bifurcations from the main cardioid, then the buried point set is arcwise connected and locally arcwise connected, and has a countable dense set of local cut points.



Bifurcations in the Main Cardiod

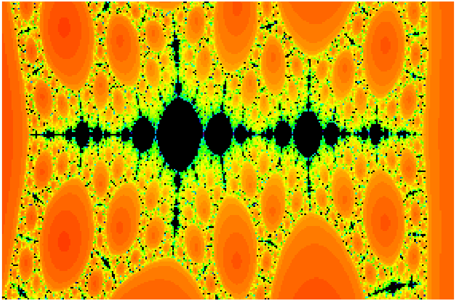


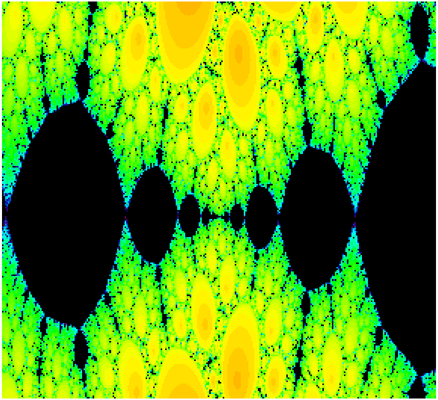
4-bulb off 2-bulb.



Embedded basillica of basillicas.

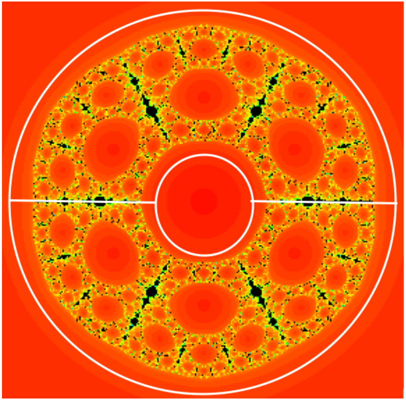
Bifurcations in the Main Cardiod



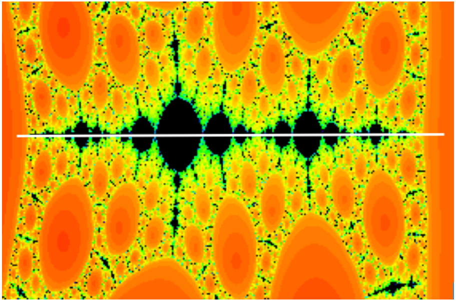


New buried points are where basillicas meet.

Bifurcations in the Main Cardiod

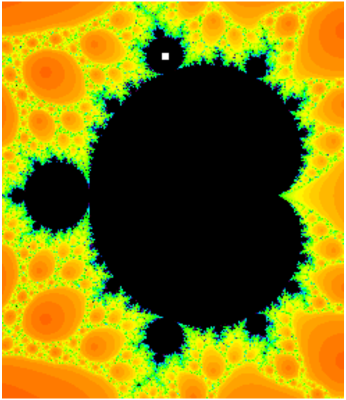


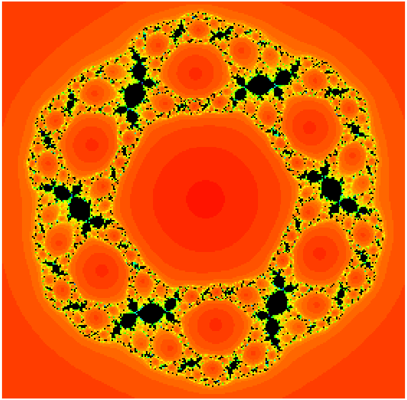
Puzzle pieces.



Ruled arcs thru quadratic Julia set.

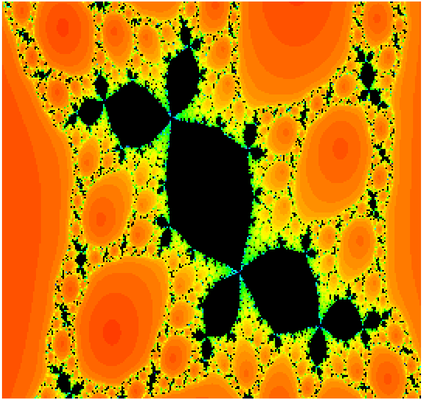
Bifurcations in the Main Cardiod





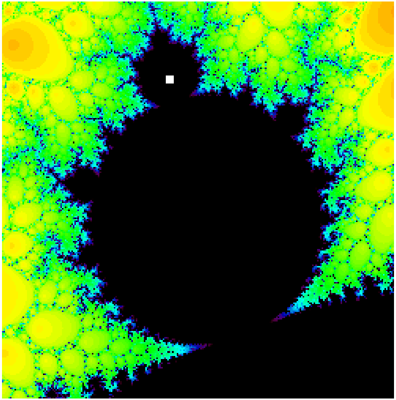
3-bulb off main cardioid. Embedded rabbit Julia sets.

Bifurcations in the Main Cardiod

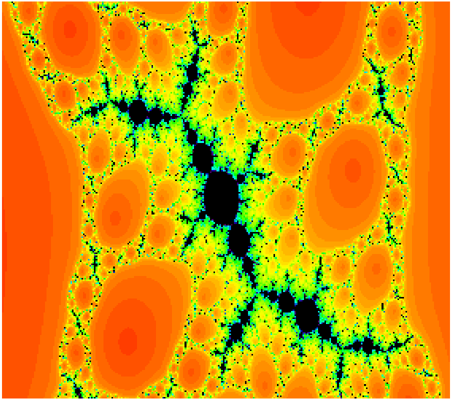


Buried point set still 0-dimensional.
Ruled arc all the way through rabbit.

Bifurcations in the Main Cardiod



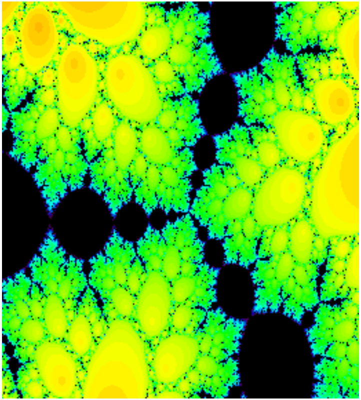
2x3-bulb off 3-bulb



Embedded rabbit of basillicas.

Bifurcations in the Main Cardiod

New buried points where three basillicas meet.



New buried points are cut points of order 3 (in the buried points).

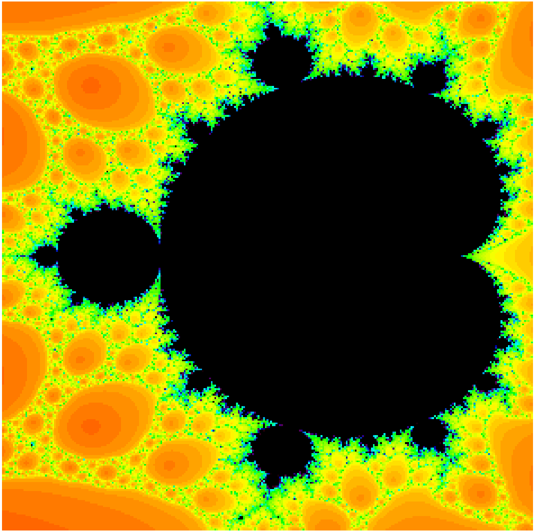
Infinitely Many Connected Buried Point Sets

Theorem

Let λ_n be in the period-doubling bulb connected to the n-bulb connected to the main cardioid.

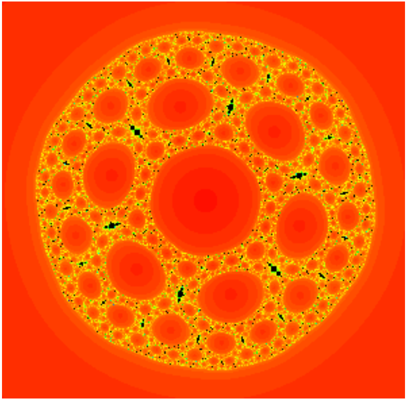
- *Then the local cut points of the set of buried points of $J(R_{\lambda_n})$ are all of order n.*
- *Hence, for $m \neq n$, the buried point sets of $J(R_{\lambda_m})$ and $J(R_{\lambda_n})$ are not homeomorphic.*

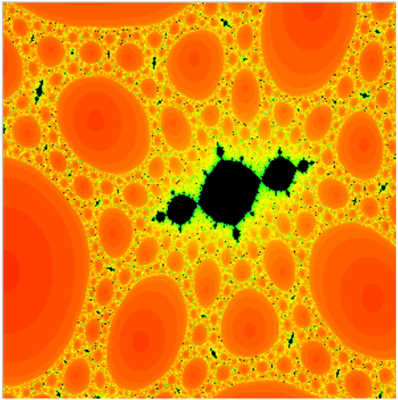
Second Bifurcations



Pairwise disjoint σ-Fatou domains

In the family $z \mapsto z^3 + \frac{\lambda}{z^3}$:
Is the buried point set homeomorphic to the buried points of the Sierpinski carpet?





Questions

Question

Is there a topological characterization of the set of buried points (the “irrational” points – Krasinkiewicz) of the Sierpinski carpet?

A conjectured start:

- Planar 1-dimensional.
- Path connected and locally path connected.
- No local separating points.
- Nowhere locally compact.
- Topologically complete.