THE UNIVERSITY of NORTH CAROLINA GREENSBORO
Department of Mathematics & Statistics

On Asymptotic Properties of Some Infinite Graph Products

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Outline

Main Idea

Antolín and Dreesen:

A finite graph product of groups with finite asymptotic dimension has finite asymptotic dimension.

Question:

To what extent does this result extend to infinite products?

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Asymptotic Dimension

Definition (Gromov)

For a metric space X, we define:
$\text{asdim}\, X \leq n \iff \forall R\, \exists \{\mathcal{U}_i\}_{i=0}^{n}$ uniformly bounded, R-disjoint families of subsets of X that cover X.

Rephrasing this:

A metric space X has $\text{asdim}\, X \leq n$ if one can paint the space with $n+1$ colors in such a way that all splotches of color have uniformly bounded diameter and so that two splotches of the same color are far apart.

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Examples with finite asdim

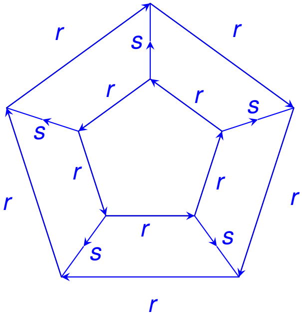

Figure : asdim $D_{2,5} = 0$.

- Compacta have asdim 0.
- asdim $\mathbb{Z} = 1$.
- asdim $\mathbb{Z}^n \leq n$.
- asdim $\mathbb{F}_2 = 1$.

Examples with finite asdim

Figure : $\operatorname{asdim} \mathbb{Z} = 1$

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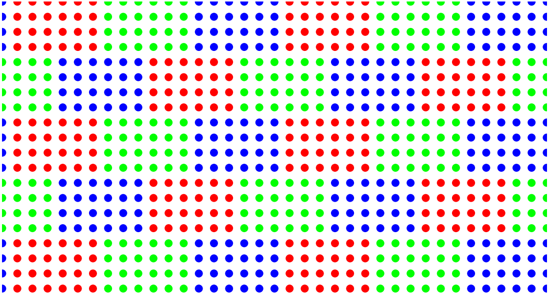

Figure : $\operatorname{asdim} \mathbb{Z}^2 \leq 2$.

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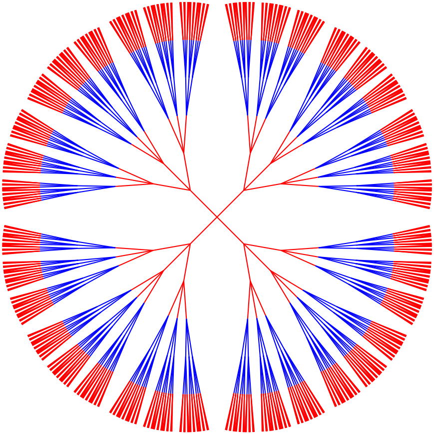

Figure : $\operatorname{asdim} \mathbb{F}_2 = 1$.

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Examples with finite asdim

These groups have infinite asdim

- Thompson's group.
- $\mathbb{Z} \wr \mathbb{Z}$
- any group containing $\mathbb{Z}^n$ for all n.

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Finitely Generated Groups

A finitely generated group can be endowed with a (left-invariant) word metric:
Given $\Gamma = \langle S \mid R \rangle$:

- For each word w in S, define $\|w\|_S$ to be the number of generators in w.
- For $g, g' \in \Gamma$, put $d(g, g') = \min\{\|w\|_S : w = g^{-1}g'\}$.
- This turns the group Γ into a proper (discrete) metric space.
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Why Large Scale Equivalent?

Consider the two presentations of $\mathbb{Z}$: $\langle a \mid \rangle$ and $\langle a, b \mid a^9 = b \rangle$.

They give rise to two Cayley graphs:

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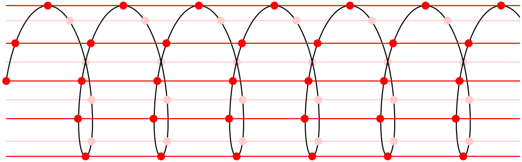

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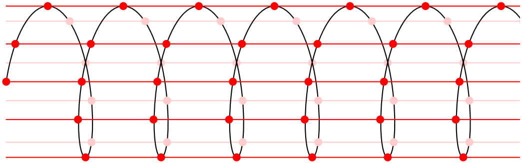

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Proper Metrics on Countable Groups

The word metric is proper, left-invariant, and any two word metrics yield equivalent metric spaces.

Question:

Given a countable group, how can we find a proper left-invariant metric that is a large-scale invariant?

Theorem (J. Smith)

1. *For countable G, $\exists!$ left-invariant, proper metric, up to coarse equivalence.*
2. *Such a metric is given by a weighting of the generating set.*

General Problem

Problem:

Given spaces (or groups) with known asdim, construct a new space and compute its asdim.

Examples

1. Direct product. (Easy)
2. Amalgamated products. (Dranishnikov)
3. Graph products. (Antolín-Dreesen)

Outline

Graph Groups

Definition (Baudisch ’70s)

A graph group is a group of the form $\langle S \mid R \rangle$ where the only permissible relations are commutators of generators.

Rephrasing:

Equivalently, take $\Gamma = (V, E)$ and put $G = \langle V \mid R \rangle$ where $R = \{[v_i, v_j] \mid (v_i, v_j) \in E\}$.

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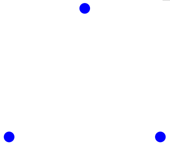
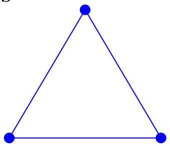


$G = \mathbb{F}_3$



$G = \mathbb{Z}^3$

Graph Products of Groups

Definition (Green 1990)

Let $\Gamma = (V, E)$ be a graph. Let $\mathfrak{G} = \{G_v : v \in V\}$ be a collection of groups. Then, the graph product is the group

$$\Gamma\mathfrak{G} = \left\langle G_k \mid [G_{v_i}, G_{v_j}], \forall (v_i, v_j) \in E \right\rangle .$$

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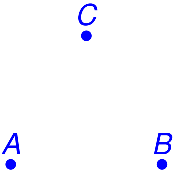

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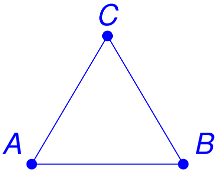

$A \times B \times C$

Two Extreme Cases

$A * B * C$ $\quad$ $A \times B \times C$

Folklore:

What holds for direct products and amalgams holds for graph products.

Outline

Antolín – Dreesen result

Lemma (Green).

If

- $\Gamma = (V, E)$ simplicial graph
- $\mathfrak{G}$ collection of groups indexed by V.

Then, $\forall\, v \in V$,

$$\Gamma\mathfrak{G} = G_A *_{G_C} G_B,$$

where $C = link_\Gamma(v)$, $B = \{v\} \cup link_\Gamma(v)$ and $A = V \setminus \{v\}$.

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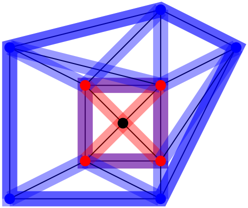

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Theorem (Dranishnikov)

1. $\operatorname{asdim} A \times B \leq \operatorname{asdim} A + \operatorname{asdim} B$.
2. $\operatorname{asdim} A *_C B \leq \max\{\operatorname{asdim} A, \operatorname{asdim} B, \operatorname{asdim} C + 1\}$

Antolín–Dreesen Result

Theorem (Antolín–Dreesen)

$\operatorname{asdim} \Gamma\mathfrak{G} \leq n$, *where*

$$n = \max\left\{\sum_{v\in C} \max\{1, \operatorname{asdim} G_v\} : C \text{ complete graph}\right\}$$

Proof.

Uses

- Induction on $|V\Gamma|$,
- Green's result decomposing the product as an amalgam
- Results on asdim of products.

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How to extend?

Some things that cause difficulty:

- Need to describe metric.
- Need to use different techniques.

Main Result

Theorem (B.-Moran)

Let Γ be a (locally finite) tree and let $\{G_v\}$ be a collection of f.g. groups with $\operatorname{asdim} G_v \leq n$. Endow $G = \Gamma\mathfrak{G}$ with a left-invariant proper metric. Then, $\operatorname{asdim} G \leq 2n$.

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Theorem (Easy special case)

Put $\Gamma = \mathbb{N}$, $G_n = \mathbb{Z}$ for all n, and set weights equal to 2^n. Then $\operatorname{asdim} G = 2$.

Greg Bell

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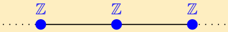
Put $\Gamma = \mathbb{N}$, $G_n = \mathbb{Z}$ *for all* n, *and set weights equal to* 2^n. *Then* $\operatorname{asdim} G = 2$.

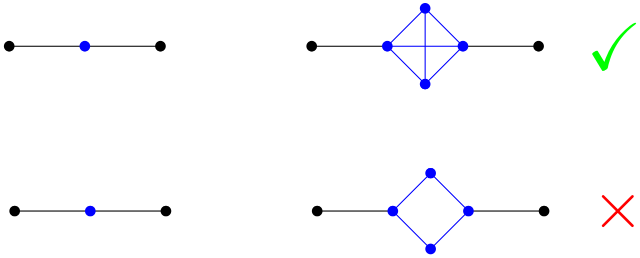
Proof.

1. Let R be given.
2. Take n s.t. $2^n \leq R < 2^{n+1}$
3. Apply Antolín-Dreesen and create covers.



Other graphs?

Our theorem also holds when Γ is obtained from a tree by replacing vertices by complete graphs.



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Other properties

The techniques used by Antolín-Dreesen can be used to show that

- (Finite) graph products preserve "property A."
- Follows from this property being preserved under free products (K. Dykema exactness; J.-L. Tu; G.B.)

There is hope that

- (Finite) graph products preserve "asymptotic property C."

Example

On the tree $\mathbb{N}$; place a copy of $\mathbb{Z}$ at all odd vertices and at $2n$ place a copy of $\mathbb{Z}^{2n}$. Give it a left-invariant proper metric, say, weighted by 2^n.

Then $\Gamma\mathfrak{G}$ has asymptotic property C.

This is similar to (but much easier than) a question of Dranishnikov–Zarichnyi (Does $\bigoplus_n Z^n$ have C?)

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Question

Suppose Γ *is a graph with uniformly bounded valence such that*

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Does it follow that $\operatorname{asdim}\Gamma\mathfrak{G} \leq n$*?*

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To what extent do infinite graph products have property A and asymptotic property C?

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