A construction of hyperbolic right-angled Coxeter groups whose boundaries are a Menger universal curve

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Motivation

It is said that N. Benakli constructed a hyperbolic Coxeter group whose boundary is a Menger universal curve.

Then I started to give an elementary and simple construction by myself, adding to interesting results.
Right-angled Coxeter groups

Definition ((Right-angled) Coxeter group and Coxeter system)

A Coxeter group is a group $W$ having a presentation

$$\langle S \mid (st)^{m(s,t)} = 1 \text{ for } s, t \in S \rangle,$$

where $S$ is a finite set and $m : S \times S \to \mathbb{N} \cup \{\infty\}$ is a function satisfying the following conditions:

1. $m(s, t) = m(t, s)$ for each $s, t \in S$,
2. $m(s, s) = 1$ for each $s \in S$, and
3. $m(s, t) \geq 2$ for each $s, t \in S$ such that $s \neq t$.

The pair $(W, S)$ is called a Coxeter system.

If, in addition,

4. $m(s, t) = 2$ or $\infty$ for each $s, t \in S$ such that $s \neq t$,

then $(W, S)$ is said to be right-angled. A group $W$ is called a right-angled Coxeter group, if there exists a generating set $S \subset W$ such that $(W, S)$ is a right-angled Coxeter system.
Nerves

Definition (Nerve of a right-angled Coxeter system)

The nerve $K$ of a right-angled Coxeter system $(W, S)$ is a finite simplicial complex defined as follows:

1. The vertex set of $K$ is the set $S$ and
2. For each subset $T$ of $S$, $T$ spans a simplex of $K$ if and only if $m(s, t) = 2$ for each $s, t \in T$ with $s \neq t$, i.e., $K$ is a flag complex.

Also a finite flag complex $K$ determines the right-angled Coxeter system $(W, S)$ with $K$ as the nerve. We only consider that $K$ is a finite simplicial complex satisfying that all the edges have length one and that it has the length metric $d_K$.

Remark (The dimension of the nerve of a right-angled Coxeter system)

Let $(W, S)$ be a right-angled Coxeter system with the nerve $K$.

1. Then, $\dim K = 1$ if and only if the length $\ell(c)$ of any circle $c$ in $K^{(1)}$ is greater than 3.

2. Then, $(W, S)$ is hyperbolic if and only if $K$ has the no- $\square$ condition i.e., for every circle $L$ in $K^{(1)}$ with 4 edges and 4 vertices, some opposite vertices in $L$ span an edge (G. Moussong).
Davis complexes

Remark(Davis complex)

(1) Every Coxeter system \((W, S)\) determines a *Davis complex* \(\Sigma = \Sigma(W, S)\) which is a CAT(0) geodesic space with its boundary \(\partial \Sigma\).

(2) \(\Sigma^{(1)}\) is the Cayley graph of \(W\) with respect to the generating set \(S\).

(3) The natural action of \(W\) on \(\Sigma\) is proper, cocompact and by isometries.

(4) We can consider a certain fundamental domain \(C\) which is called a *chamber* of \(\Sigma\) such that \(WC = \Sigma\). Here we can identify the chamber \(C\) as the cone of the nerve \(K\).

(5) Let \(B(n) = \bigcup \{aC \mid a \in W, \ell_S(a) \leq n\}\) and let \(S(n)\) be the boundary of \(B(n)\) in \(\Sigma\) for each \(n \in \mathbb{N}\). Then, there exists a natural projection \(\rho_n^{n+1} : S(n + 1) \rightarrow S(n)\) such that \(\partial \Sigma\) is homeomorphic to \(\lim\{S(n), \rho_n^{n+1}\}\).
Definition

A connected simplicial complex \((K, d_K)\) is said to be *strongly co-connected* if \(\{y \in K \mid d_K(x, y) \geq 2\}\) is a nonempty connected set for each \(x \in X\).

Definition

A connected simplicial complex \(K\) is said to have no cut pair, if \(K \setminus \{x, y\}\) is a nonempty connected set for any \(x, y\) in \(K\) satisfying that no simplex of \(K\) contains \(\{x, y\}\).
Main results

The following theorem provides a criterion for boundaries which are homeomorphic to either a Sierpiński carpet or a Menger universal curve.

**Main Theorem (C-Hosaka)**

Let $K$ be a strongly co-connected finite simplicial 1-complex, let $\Sigma$ be the Davis complex of the right-angled Coxeter system $(W, S)$ with the nerve $K$, and let $\partial \Sigma$ be the boundary of $\Sigma$.

1. Then, $\partial \Sigma$ is homeomorphic to a Sierpiński carpet if and only if $K$ has no cut pair and $K \hookrightarrow S^2$.
2. Then, $\partial \Sigma$ is homeomorphic to a Menger universal curve if and only if $K$ has no cut pair and $K \not\hookrightarrow S^2$.

Using main theorem, we construct concrete examples of hyperbolic right-angled Coxeter groups with boundaries as a Sierpiński carpet and a Menger universal curve.
Construction

Definition

A connected simplicial complex \((K, d_K)\) is said to be **strongly co-connected** if \(\{y \in K \mid d_K(x, y) \geq 2\}\) is a nonempty connected set for each \(x \in X\).

Definition

A connected simplicial complex \(K\) is said to **have no cut pair**, if \(K \setminus \{x, y\}\) is a nonempty connected set for any \(x, y\) in \(K\) satisfying that no simplex of \(K\) contains \(\{x, y\}\).

Remark

Let \(K\) be a 1-dimensional strongly co-connected simplicial complex. Then, \(K\) is a flag complex.

Remark

Let \(K\) be a 1-dimensional strongly co-connected simplicial complex with no cut pair and let \((W, S)\) be the right-angled Coxeter system with the nerve \(K\). Then, \(W\) is hyperbolic.
Then, $F$ and $T_{16}$ are strongly co-connected finite simplicial 1-complexes with no cut pair. Let $(W_0, S_0)$ and $(W_1, S_1)$ be the hyperbolic right-angled Coxeter systems with $F$ and $T_{16}$ as the nerves, respectively. From main theorem, $\partial W_0$ is homeomorphic to a Sierpiński carpet and $\partial W_1$ is homeomorphic to a Menger universal curve.
Then, $R_6$ has no cut pair, but not strongly co-connected
\[
\{ y \in K \mid d_K(x, y) \geq 2 \} \text{ is not connected}, \text{ and}
\]
$F_{2,2}$ is strongly co-connected, but has a cut pair.
Definition

Let $L$ be a 2-skeleton of a connected closed PL $n$-manifold $M$ with $n \geq 2$ and let $F$ be a truncated icosahedron as above. Fix a hexagon $H$ in the set of all 2-cells of $F$. Set $D = \text{Cl}_F(F \setminus H)$. We replace of all 2-simplexes of $L$ by copies of $D$ as follows: For every 2-simplex $\sigma$ of $L$, let $D_\sigma$ be a copy of $D$ such that $\text{Int}D_\sigma \cap \text{Int}D_{\sigma'} = \emptyset$ whenever $\sigma \neq \sigma'$. For every 2-simplex $\sigma$ of $L$, we can identify $(\text{sd}(\sigma^{(1)}), \{\text{sd}(\sigma^{(1)})^{(0)}\})$ with $(\partial D_\sigma, (\partial D_\sigma)^{(0)})$, and, set $L_F = \text{sd}(L^{(1)}) \cup \bigcup \{D_\sigma | \sigma \text{ is a 2-simplex of } L\}$ with the natural cell subdivision.
We can show that $L^{(1)}_F$ is strongly co-connected with no cut pair. Hence,

**Theorem (C-Hosaka)**

Let $L$, $M$, and $L_F$ be as above, and, let $(W, S)$ be the hyperbolic right-angled Coxeter system with $L^{(1)}_F$ as the nerve.

1. Then, $\partial W$ is homeomorphic to a Sierpiński carpet if and only if $M$ is homeomorphic to $S^2$.
2. Then, $\partial W$ is homeomorphic to a Menger universal curve if and only if $M$ is not homeomorphic to $S^2$. 
(Sketch of proof of Main Theorem)

Let $K$ be a strongly co-connected finite simplicial 1-complex, let $\Sigma$ be the Davis complex of the right-angled Coxeter system $(W, S)$ with the nerve $K$.

We use the characterizations of a Sierpiński carpet due to G. T. Whyburn, and a Menger universal curve due to R. D. Anderson.

(Step 1)
Let $m, n \in \mathbb{N}$ with $m > n$ and $w \in W$ with $\ell_S(w) = n + 1$. We show that $(\rho_m^n)^{-1}(wK \cap S(n))$ is connected. (Note that a fiber of a projection $\rho_m^n : S(m) \to S(n)$ is not necessarily connected.)

(Step 2)
By Step 1, $\partial \Sigma$ has no local cut point if and only if $K$ has no cut pair.

(Step 3)
By Steps 1 and 2, for every open subset $U$ of $\partial W$, there exists a finite graph $K' \hookrightarrow U$ which contracts to $K$. 