A construction of hyperbolic right-angled Coxeter groups whose boundaries are a Menger universal curve

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Motivation

It is said that N. Benakli constructed
a hyperbolic Coxeter group whose boundary is a Menger universal curve.

Then I started to give an elementary and simple construction by myself, adding to interesting results.

Right-angled Coxeter groups

Definition ((Right-angled) Coxeter group and Coxeter system)

A *Coxeter group* is a group W having a presentation

$$\langle S \mid (st)^{m(s,t)} = 1 \text{ for } s, t \in S \rangle,$$

where S is a finite set and $m : S \times S \to \mathbb{N} \cup \{\infty\}$ is a function satisfying the following conditions:

(1) $m(s,t) = m(t,s)$ for each $s, t \in S$,

(2) $m(s,s) = 1$ for each $s \in S$, and

(3) $m(s,t) \geq 2$ for each $s, t \in S$ such that $s \neq t$.

The pair (W, S) is called a *Coxeter system*.

If, in addition,

(4) $m(s,t) = 2$ or ∞ for each $s, t \in S$ such that $s \neq t$,

then (W, S) is said to be *right-angled*. A group W is called a *right-angled Coxeter group*, if there exists a generating set $S \subset W$ such that (W, S) is a right-angled Coxeter system.

Nerves

Definition (Nerve of a right-angled Coxeter system)

The *nerve* K of a right-angled Coxeter system (W, S) is a finite simplicial complex defined as follows:

(1) the vertex set of K is the set S and

(2) for each subset T of S, T spans a simplex of K if and only if $m(s,t) = 2$ for each $s, t \in T$ with $s \neq t$, i.e., K is a flag complex.

Also a finite flag complex K determines the right-angled Coxeter system (W, S) with K as the nerve. We only consider that K is a finite simplicial complex satisfying that all the edges have length one and that it has the length metric d_K.

Remark(The dimension of the nerve of a right-angled Coxeter system)

Let (W, S) be a right-angled Coxeter system with the nerve K.

(1) Then, $\dim K = 1$ if and only if the length $\ell(c)$ of any circle c in $K^{(1)}$ is greater than 3.

(2) Then, (W, S) is hyperbolic if and only if K has the no-$\square$ condition i.e., for every circle L in $K^{(1)}$ with 4 edges and 4 vertices, some opposite vertices in L span an edge (G. Moussong).

Davis complexes

Remark(Davis complex)

(1) Every Coxeter system (W, S) determines a *Davis complex* $\Sigma = \Sigma(W, S)$ which is a CAT(0) geodesic space with its boundary $\partial\Sigma$.

(2) $\Sigma^{(1)}$ is the Cayley graph of W with respect to the generating set S.

(3) The natural action of W on Σ is proper, cocompact and by isometries.

(4) We can consider a certain fundamental domain C which is called a *chamber* of Σ such that $WC = \Sigma$. Here we can identify the chamber C as the cone of the nerve K.

(5) Let $B(n) = \bigcup\{aC \mid a \in W, \ell_S(a) \leq n\}$ and let $S(n)$ be the boundary of $B(n)$ in Σ for each $n \in \mathbb{N}$. Then, there exists a natural projection $\rho_n^{n+1} : S(n+1) \to S(n)$ such that $\partial\Sigma$ is homeomorphic to $\varprojlim\{S(n), \rho_n^{n+1}\}$.

Definition

A connected simplicial complex (K, d_K) is said to be *strongly co-connected* if $\{y \in K \mid d_K(x, y) \geq 2\}$ is a nonempty connected set for each $x \in X$.

Definition

A connected simplicial complex K is said to *have no cut pair*, if $K \setminus \{x, y\}$ is a nonempty connected set for any x, y in K satisfying that no simplex of K contains $\{x, y\}$.

Main results

The following theorem provides a criterion for boundaries which are homeomorphic to either a Sierpiński carpet or a Menger universal curve.

Main Theorem (C-Hosaka)

Let K be a strongly co-connected finite simplicial 1-complex, let Σ be the Davis complex of the right-angled Coxeter system (W, S) with the nerve K, and let $\partial\Sigma$ be the boundary of Σ.

(1) Then, $\partial\Sigma$ is homeomorphic to a Sierpiński carpet if and only if K has no cut pair and $K \hookrightarrow \mathbb{S}^2$.

(2) Then, $\partial\Sigma$ is homeomorphic to a Menger universal curve if and only if K has no cut pair and $K \not\hookrightarrow \mathbb{S}^2$.

Using main theorem, we construct concrete examples of hyperbolic right-angled Coxeter groups with boundaries as a Sierpiński carpet and a Menger universal curve.

Construction

Definition

A connected simplicial complex (K, d_K) is said to be *strongly co-connected* if $\{y \in K \mid d_K(x, y) \geq 2\}$ is a nonempty connected set for each $x \in X$.

Definition

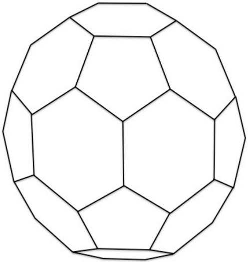
A connected simplicial complex K is said to *have no cut pair*, if $K \setminus \{x, y\}$ is a nonempty connected set for any x, y in K satisfying that no simplex of K contains $\{x, y\}$.

Remark

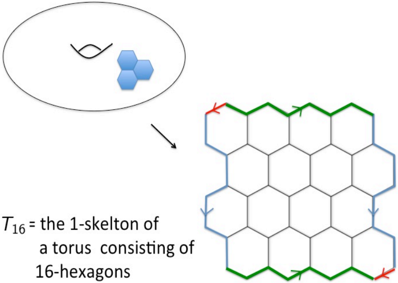
Let K be a 1-dimensional strongly co-connected simplicial complex. Then, K is a flag complex.

Remark

Let K be a 1-dimensional strongly co-connected simplicial complex with no cut pair and let (W, S) be the right-angled Coxeter system with the nerve K. Then, W is hyperbolic.

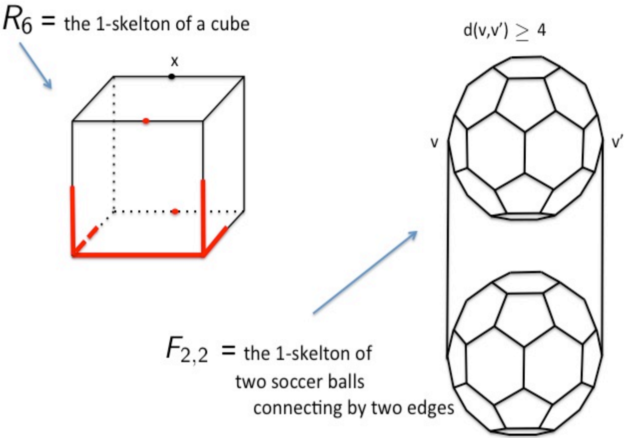


F = the 1-skelton of a truncated icosahedron



T_{16} = the 1-skelton of a torus consisting of 16-hexagons

Then, F and T_{16} are strongly co-connected finite simplicial 1-complexes with no cut pair.
Let (W_0, S_0) and (W_1, S_1) be the hyperbolic right-angled Coxeter systems with F and T_{16} as the nerves, respectively. From main theorem, ∂W_0 is homeomorphic to a Sierpiński carpet and ∂W_1 is homeomorphic to a Menger universal curve.



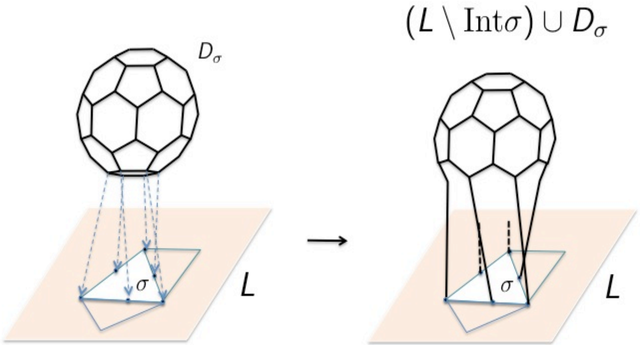
Then, R_6 has no cut pair, but not strongly co-connected

($\because \{y \in K \mid d_K(x, y) \geq 2\}$ is not connected), and

$F_{2,2}$ is strongly co-connected, but has a cut pair.

Definition

Let L be a 2-skeleton of a connected closed PL n-manifold M with $n \geq 2$ and let F be a truncated icosahedron as above. Fix a hexagon H in the set of all 2-cells of F. Set $D = \mathrm{Cl}_F(F \setminus H)$. We replace of all 2-simplexes of L by copies of D as follows: For every 2-simplex σ of L, let D_σ be a copy of D such that $\mathrm{Int}D_\sigma \cap \mathrm{Int}D_{\sigma'} = \emptyset$ whenever $\sigma \neq \sigma'$. For every 2-simplex σ of L, we can identify $(\mathrm{sd}(\sigma^{(1)}), \{\mathrm{sd}(\sigma^{(1)})\}^{(0)})$ with $(\partial D_\sigma, (\partial D_\sigma)^{(0)})$, and, set $L_F = \mathrm{sd}(L^{(1)}) \cup \bigcup\{D_\sigma \mid \sigma \text{ is a 2-simplex of } L\}$ with the natural cell subdivision.



We can show that $L_F^{(1)}$ is strongly co-connected with no cut pair. Hence,

Theorem (C-Hosaka)

Let L, M, and L_F be as above, and, let (W, S) be the hyperbolic right-angled Coxeter system with $L_F^{(1)}$ as the nerve.

(1) *Then, ∂W is homeomorphic to a Sierpiński carpet if and only if M is homeomorphic to $\mathbb{S}^2$.*

(2) *Then, ∂W is homeomorphic to a Menger universal curve if and only if M is not homeomorphic to $\mathbb{S}^2$.*

(Sketch of proof of Main Theorem)

Let K be a strongly co-connected finite simplicial 1-complex, let Σ be the Davis complex of the right-angled Coxeter system (W, S) with the nerve K.

We use the characterizations of a Sierpiński carpet due to G. T. Whyburn, and a Menger universal curve due to R. D. Anderson.

(Step 1)
Let $m, n \in \mathbb{N}$ with $m > n$ and $w \in W$ with $\ell_S(w) = n + 1$. We show that $(\rho_n^m)^{-1}(wK \cap S(n))$ is connected. (Note that a fiber of a projection $\rho_n^m : S(m) \to S(n)$ is not necessarily connected.)

(Step 2)
By Step 1, $\partial\Sigma$ has no local cut point if and only if K has no cut pair.

(Step 3)
By Steps 1 and 2, for every open subset U of ∂W, there exists a finite graph $K' \hookrightarrow U$ which contracts to K.