HYPERSPACES OF KELLER COMPACTA AND THEIR ORBIT SPACES

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National University of Mexico (UNAM)
1. Keller compacta

2. G-spaces

3. The problem and the result

4. Important notions

5. Sketch of the proof
Keller compacta

An infinite-dimensional compact convex subset $K$ of a topological linear space is called a Keller compactum, if it is affinely embeddable in the Hilbert space $\ell_2$:

$$K \hookrightarrow \ell_2 = \{(x_n) \mid x_n \in \mathbb{R}, \sum_{n=1}^{\infty} x_n^2 < \infty\}.$$
Let $K$ and $V$ be convex subsets of linear spaces.

A map $f : K \rightarrow V$ is called \textbf{affine}, if for every $x_1, \ldots, x_n \in K$ and $t_1, \ldots, t_n \in [0, 1]$ such that $\sum_{i=1}^{n} t_i = 1$

$$f \left( \sum_{i=1}^{n} t_i x_i \right) = \sum_{i=1}^{n} t_i f(x_i).$$
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The Hilbert cube

\[ Q = \prod_{n=1}^{\infty} [-1, 1]_n \subset \mathbb{R}^\infty \]

is affinely homeomorphic to \( \{ x \in \ell_2 \mid |x_n| \leq 1/n \} \subset \ell_2 \).
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The space \( P(X) \) of probability measures of an infinite compact metric space \( X \) endowed with the topology of weak convergence in measures:

\[ \mu_n \overset{\sim}{\to} \mu \iff \int f \, d\mu_n \overset{\sim}{\to} \int f \, d\mu \quad \forall f \in C(X). \]
Theorem (O. H. Keller)

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However, not all Keller compacta are affinely homeomorphic to each other.

We consider Keller compacta together with its affine-topological structure.
Let $G$ be a topological group. A **G-space** is a topological space $X$ together with a fixed continuous action of $G$:

\[ G \times X \longrightarrow X, \quad (g, x) \longmapsto gx. \]
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A map $f : X \rightarrow Y$ between $G$-spaces is called **equivariant** if for every $x \in X$ and $g \in G$,

\[ f(gx) = gf(x) \]
Let $X$ be a $G$-space. A subset $A \subset X$ is called **invariant** if

$$A = \{ga \mid g \in G, \ a \in A\}.$$
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The **orbit of** $x \in X$ is the smallest invariant subset containing $x$:

$$Gx = \{ gx \mid g \in G \}.$$
The orbit space of $X$ is the set

$$X/G = \{ Gx \mid x \in X \}$$

endowed with the quotient topology given by the orbit map

$$X \longrightarrow X/G, \quad x \longmapsto Gx.$$
Let \((X,d)\) be a metric space. The **hyperspace** of \(X\):

\[2^X = \{ A \subset X | \emptyset \neq A \text{ compact}\}\]

endowed with the topology induced by the Hausdorff metric:

\[d_H(A,B) = \max \left\{ \sup_{b \in B} d(b,A), \sup_{a \in A} d(a,B) \right\}, \quad A, B \in 2^X.\]
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\]

Let \(X\) be a subset of a topological linear space. The **cc-hyperspace** of \(X\):

\[
cc(X) = \{ A \in 2^X \mid A \text{ convex} \}.
\]
If $X$ is a metrizable $G$-space, then $2^X$ becomes a $G$-space with the induced action:

$$G \times 2^X \longrightarrow 2^X, \quad (g, A) \longmapsto gA = \{ga \mid a \in A\}.$$ 

In case $X$ is a subset of a topological linear space and every $g \in G$ preserves convexity, $cc(X)$ is an invariant subspace of $2^X$ under this action.
**Question (J. West, 1976)**

Let $G$ be a compact connected Lie group. Is the orbit space $2^G/G$ an AR? If it is, is it homeomorphic to the Hilbert cube $Q$?
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**Motivation**

**Question (J. West, 1976)**

*Let G be a compact connected Lie group. Is the orbit space $2^G/G$ an AR? If it is, is it homeomorphic to the Hilbert cube $Q$?*

Toruńczyk and West proved that $2^{S^1}/S^1 \in \text{AR}$ not homeomorphic to $Q$.

Antonyan proved that $2^{S^1}/O(2) \cong BM(2)$, which is an AR but not homeomorphic to $Q$. 
**Theorem (S. Antonyan)**

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For \( n \geq 2 \), the orbit space \( cc(\mathbb{B}^n) / O(n) \) is homeomorphic to \( \text{cone}(BM(n)) \).

The Hilbert cube \( Q \) is a natural infinite-dimensional analog of \( \mathbb{B}^n \). An analog for \( O(n) \) is the group \( O(Q) \) of affine isometries of \( Q \), which leave the origin fixed.
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The Hilbert cube $Q$ is a natural infinite-dimensional analog of $\mathbb{B}^n$. An analog for $O(n)$ is the group $O(Q)$ of affine isometries of $Q$, which leave the origin fixed.

The purpose of this talk is to show that

$$2^Q/O(Q) \cong Q \quad \text{and} \quad cc(Q)/O(Q) \cong Q.$$
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In analogy to the action of $O(n)$ in $\mathbb{B}^n$, we consider actions of compact groups on centrally symmetric Keller compacta that respect their affine-topological structure.
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In analogy to the action of $O(n)$ in $\mathbb{B}^n$, we consider actions of compact groups on centrally symmetric Keller compacta that respect their affine-topological structure.

We say that a group $G$ acts **affinely** on a Keller compactum $K$ if for every $g \in G$, $x_1, \ldots, x_n \in K$ and $t_1, \ldots, t_n \in [0,1]$ such that $\sum_{i=1}^n t_i = 1$

$$g\left(\sum_{i=1}^n t_i x_i\right) = \sum_{i=1}^n t_i g x_i.$$
**Problem**

*Given a centrally symmetric Keller compactum $K$ (e.g., the Hilbert cube $Q \subset \mathbb{R}^\infty$), describe the topological structure of the orbit spaces of $2^K$ and $cc(K)$ with respect to the affine action of a compact group $G$ (not necessarily Lie).*
**Problem**

Given a centrally symmetric Keller compactum $K$ (e.g., the Hilbert cube $Q \subset \mathbb{R}^\infty$), describe the topological structure of the orbit spaces of $2^K$ and $cc(K)$ with respect to the affine action of a compact group $G$ (not necessarily Lie).

**Theorem**

Let $G$ be a compact group acting affinely on a centrally symmetric Keller compactum $K$, then the orbit spaces $2^K/G$ and $cc(K)/G$ are homeomorphic to $Q$. 
Let $K$ be a Keller compactum. A point $x_0 \in K$ is called a **center of symmetry**, if for every $x \in K$, there is a point $y \in K$ such that

$$x_0 = (x + y)/2.$$ 

If $K$ admits a center of symmetry, then it is called **centrally symmetric**.
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Let $K$ be a Keller compactum. A point $x_0 \in K$ is called \textit{radially internal} if for every $x \in K$

$$\inf_{t \in \mathbb{R}} \{ |t| : x_0 + t(x - x_0) \notin K \} > 0.$$
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$$\exists \ t_1 < 0, \ y = x_0 + t_1(x - x_0) \in K,$$
The **radial interior** of \( K \) is the set

\[
\text{rint } K = \{ x \in K \mid x \text{ is radially internal} \}.
\]

The **radial boundary** of \( K \) is the complement

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\text{rbd } K = K \setminus \text{rint } K.
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**Proposition**

Let $K$ and $V$ be Keller compacta and $h : K \to V$ and affine homeomorphism. Then $h(\text{rint } K) = \text{rint } V$. 
The space of probability measures $P([0,1])$ of $[0,1]$ is a Keller compactum with

$$rint P([0,1]) = \emptyset.$$ 

Since

$$rint Q = \{ x \in Q \mid \sup_{n \in \mathbb{N}} |x_n| < 1 \} \neq \emptyset$$

$P([0,1])$ cannot be affinely-homeomorphic to $Q$. 
Let \((X, d)\) be a metric \(G\)-space. If for every \(x, y \in X\) and \(g \in G\),
\[
d(gx, gy) = d(x, y),
\]
then we say that \(d\) is an **invariant metric** and that the action of \(G\) is **isometric**.
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If \(G\) is compact, then every metric \(G\)-space \(X\) admits an invariant metric \(d\). In this situation, \(d\) induces a metric in the orbit space \(X/G\):

\[
d^*(Gx, Gy) = \inf \{ d(gx, g'y) \mid g, g' \in G \}.
\]
A metrizable $G$-space $X \in \mathbf{G-ANR}$, if for every metrizable $G$-space $Y$ containing $X$ as a closed invariant subset, there is an invariant neighborhood $U$ of $X$ in $Y$ and a $G$-retraction $r : U \to X$.

If we can always take $U = Y$, then we say that $X \in \mathbf{G-AR}$.
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\[
\begin{array}{c}
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\bigcirc \\
U \longrightarrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad X
\end{array}
\]
Theorem (S. Antonyan)

Let $G$ be a compact group and $X \in G\text{-ANR}$ (resp., $G\text{-AR}$). Then the orbit space $X/G \in \text{ANR}$ (resp., AR).

Theorem (S. Antonyan)

Let $G$ be a compact group and $X$ a completely metrizable locally connected $G$-space. Then $2^X$ is a $G\text{-ANR}$. If, in addition, $X$ is connected, then $2^X$ is a $G\text{-AR}$.
A **$Q$-manifold** is a separable metrizable space that is locally homeomorphic to the Hilbert cube $Q$.

**Teorema (H. Toruńczyk)**

A locally compact ANR $X$ is a $Q$-manifold if and only if for every $\epsilon > 0$ there exist continuous maps $f_1, f_2 : X \to X$ such that $d(f_1, 1_X) < \epsilon$, $d(f_2, 1_X) < \epsilon$ and $\text{im}(f_1) \cap \text{im}(f_2) = \emptyset$. 
Result

Sketch of the proof:

The orbit space $cc(K)/G$ is a compact AR. Let $\epsilon > 0$ and $x_0$ the center of symmetry. We construct equivariant maps

$$f_1 : cc(K) \to cc(K) \quad \text{and} \quad f_2 : cc(K) \to cc(K)$$

$\epsilon$-close to the identity map $1_{cc(K)}$ such that

$$im f_1 \subset rint K \quad \text{and} \quad im f_2 \cap rbd K \neq \emptyset.$$
Indeed,

\[ f_1(A) = x_0 + t(A - x_0), \quad A \in cc(K), \quad t \in (1 - \epsilon, 1) \]

\[ f_2(A) = \{ x \in K \mid d(x, A) \leq \epsilon \}, \quad A \in cc(K). \]

Then \( f_1 \) and \( f_2 \) induce continuous maps

\[ \tilde{f}_1 : cc(K)/G \to cc(K)/G \quad \text{and} \quad \tilde{f}_2 : cc(K)/G \to cc(K)/G \]

also satisfying the properties of Toruńczyk’s Theorem. \( \square \)
**Theorem**

Let $G$ be a compact group acting affinely on a Keller compactum $K$. If there is a $G$-fixed point $x_0 \in \text{rint } K$, then the orbit spaces $2^K/G$ and $cc(K)/G$ are homeomorphic to the Hilbert cube $Q$. 
THE END