Hyperspaces of Keller compacta and their orbit spaces

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Joint work with Sergey Antonyan and Natalia Jonard-Pérez.

National University of Mexico (UNAM)

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Keller compacta

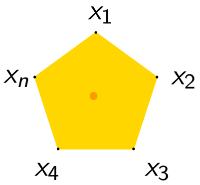
An infinite-dimensional compact convex subset K of a topological linear space is called a **Keller compactum**, if it is affinely embeddable in the Hilbert space ℓ_2:

$$K \hookrightarrow \ell_2 = \{(x_n) \mid x_n \in \mathbb{R}, \sum_{n=1}^{\infty} x_n^2 < \infty\}.$$

Let K and V be convex subsets of linear spaces.

A map $f : K \to V$ is called **affine**, if for every $x_1, \ldots, x_n \in K$ and $t_1, \ldots, t_n \in [0, 1]$ such that $\sum_{i=1}^n t_i = 1$

$$f\Big(\sum_{i=1}^{n} t_i x_i\Big) = \sum_{i=1}^{n} t_i f(x_i).$$



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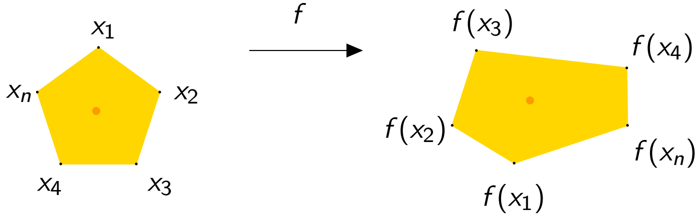

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$$Q = \prod_{n=1}^{\infty} [-1, 1]_n \subset \mathbb{R}^\infty$$

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The space $P(X)$ of probability measures of an infinite compact metric space X endowed with the topology of weak convergence in measures:

$$\mu_n \rightsquigarrow \mu \qquad \Longleftrightarrow \qquad \int f\, d\mu_n \rightsquigarrow \int f\, d\mu \qquad \forall f \in C(X).$$

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However, not all Keller compacta are affinely homeomorphic to each other.

We consider Keller compacta together with its affine-topological structure.

G-SPACES

Let G be a topological group. A ***G*-space** is a topological space X together with a fixed continuous action of G:

$$G \times X \longrightarrow X, \qquad (g,x) \longmapsto gx.$$

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Let G be a topological group. A ***G*-space** is a topological space X together with a fixed continuous action of G:

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A map $f : X \to Y$ between G-spaces is called **equivariant** if for every $x \in X$ and $g \in G$,

$$f(gx) = gf(x)$$

$$\begin{array}{ccc} G \times X & \longrightarrow & X \\ \Big\downarrow{\scriptstyle 1\times f} & & \Big\downarrow{\scriptstyle f} \\ G \times Y & \longrightarrow & Y \end{array}$$

Let X be a G-space. A subset $A \subset X$ is called **invariant** if

$$A = \{ga \mid g \in G,\ a \in A\}.$$

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The **orbit of x** $\in X$ is the smallest invariant subset containing x:

$$Gx = \{gx \mid g \in G\}.$$

The **orbit space** of X is the set

$$X/G = \{Gx \mid x \in X\}$$

endowed with the quotient topology given by the **orbit map**

$$X \longrightarrow X/G, \qquad x \longmapsto Gx.$$

Let (X,d) be a metric space. The **hyperspace** of X:

$$2^X = \{A \subset X \mid \emptyset \neq A \text{ compact}\}$$

endowed with the topology induced by the Hausdorff metric:

$$d_H(A,B) = \max\left\{\sup_{b\in B} d(b,A), \sup_{a\in A} d(a,B)\right\}, \quad A,B \in 2^X.$$

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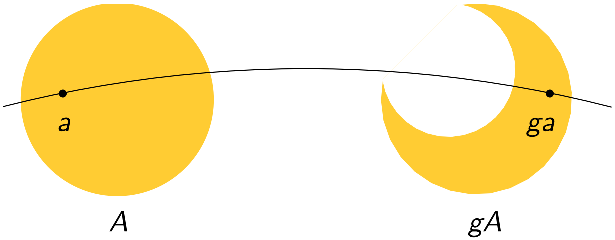
Let X be a subset of a topological linear space. The **cc-hyperspace** of X:

$$cc(X) = \{A \in 2^X \mid A \text{ convex}\}.$$

If X is a metrizable G-space, then 2^X becomes a G-space with the induced action:

$$G \times 2^X \longrightarrow 2^X, \qquad (g, A) \longmapsto gA = \{ga \mid a \in A\}.$$

In case X is a subset of a topological linear space and every $g \in G$ preserves convexity, $cc(X)$ is an invariant subspace of 2^X under this action.



Motivation

Question (J. West, 1976)

Let G be a compact connected Lie group. Is the orbit space $2^G/G$ an AR? If it is, is it homeomorphic to the Hilbert cube Q?

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Toruńczyk and West proved that $2^{\mathbb{S}^1}/\mathbb{S}^1 \in$ AR not homeomorphic to Q.

Antonyan proved that $2^{\mathbb{S}^1}/O(2) \cong BM(2)$, which is an AR but not homeomorphic to Q.

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The Hilbert cube Q is a natural infinite-dimensional analog of $\mathbb{B}^n$.

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An analog for $O(n)$ is the group $O(Q)$ of affine isometries of Q, which leave the origin fixed.

The purpose of this talk is to show that

$$2^Q/O(Q) \cong Q \qquad cc(Q)/O(Q) \cong Q.$$

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We say that a group G acts **affinely** on a Keller compactum K if for every $g \in G$, $x_1, \dots, x_n \in K$ and $t_1, \dots, t_n \in [0,1]$ such that $\sum_{i=1}^n t_i = 1$

$$g\Big(\sum_{i=1}^{n} t_i x_i\Big) = \sum_{i=1}^{n} t_i g x_i.$$

Problem

Given a centrally symmetric Keller compactum K (e.g., the Hilbert cube $Q \subset \mathbb{R}^\infty$), describe the topological structure of the orbit spaces of 2^K and $cc(K)$ with respect to the affine action of a compact group G (not necessarily Lie).

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Theorem

Let G be a compact group acting affinely on a centrally symmetric Keller compactum K, then the orbit spaces $2^K/G$ and $cc(K)/G$ are homeomorphic to Q.

Let K be a Keller compactum. A point $x_0 \in K$ is called a **center of symmetry**, if for every $x \in K$, there is a point $y \in K$ such that

$$x_0 = (x + y)/2.$$

If K admits a center of symmetry, then it is called **centrally symmetric**.

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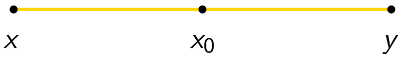

Let K be a Keller compactum. A point $x_0 \in K$ is called **radially internal** if for every $x \in K$

$$\inf_{t\in\mathbb{R}}\{|t| \mid x_0 + t(x - x_0) \notin K\} > 0.$$

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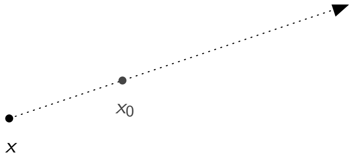

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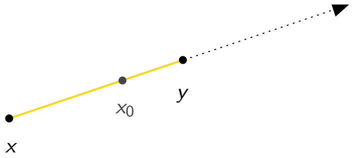

$$\exists\ t_1 < 0, \quad y = x_0 + t_1(x - x_0) \in K,$$

The **radial interior** of K is the set

$$\mathit{rint}\ K = \{x \in K \mid x \text{ is radially internal}\}.$$

The **radial boundary** of K is the complement

$$\mathit{rbd}\ K = K \setminus \mathit{rint}\ K.$$

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PROPOSITION

Let K and V be Keller compacta and $h : K \to V$ and affine homeomorphism. Then $h(rint\ K) = rint\ V$.

The space of probability measures $P([0,1])$ of $[0,1]$ is a Keller compactum with

$$rint\ P([0,1]) = \emptyset.$$

Since

$$rint\ Q = \{x \in Q \mid \sup_{n\in\mathbb{N}} |x_n| < 1\} \neq \emptyset$$

$P([0,1])$ cannot be affinely-homeomorphic to Q.

Let (X, d) be a metric G-space. If for every $x, y \in X$ and $g \in G$,

$$d(gx, gy) = d(x, y),$$

then we say that d is an **invariant metric** and that the action of G is **isometric**.

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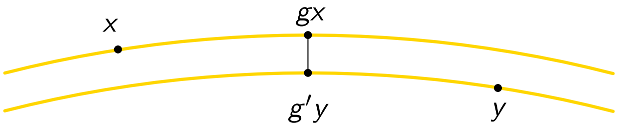
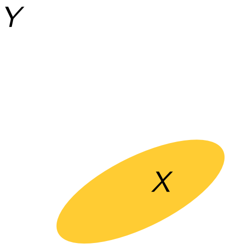
If G is compact, then every metric G-space X admits an invariant metric d. In this situation, d induces a metric in the orbit space X/G:

$$d^*(Gx, Gy) = inf\{d(gx, g'y) \mid g, g' \in G\}.$$



A metrizable G-space $X \in$ ***G*-ANR**, if for every metrizable G-space Y containing X as a closed invariant subset, there is an invariant neighborhood U of X in Y and a G-retraction $r : U \to X$.

If we can always take $U = Y$, then we say that $X \in$ ***G*-AR**.



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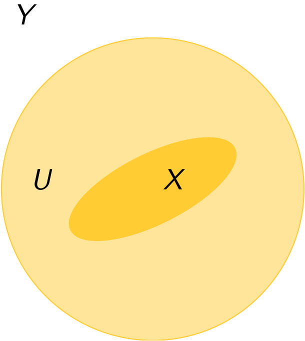

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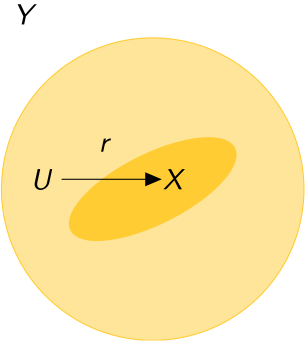

Theorem (S. Antonyan)

Let G be a compact group and $X \in G$-ANR (resp., G-AR). Then the orbit space $X/G \in$ ANR (resp., AR).

Theorem (S. Antonyan)

Let G be a compact group and X a completely metrizable locally connected G-space. Then 2^X is a G-ANR. If, in addition, X is connected, then 2^X is a G-AR.

A ***Q*-manifold** is a separable metrizable space that is locally homeomorphic to the Hilbert cube Q.

Teorema (H. Toruńczyk)

A locally compact ANR X *is a* Q*-manifold if and only if for every* $\epsilon > 0$ *there exist continuous maps* $f_1, f_2 : X \to X$ *such that* $d(f_1, 1_X) < \epsilon$, $d(f_2, 1_X) < \epsilon$ *and* $im(f_1) \cap im(f_2) = \emptyset$.

Result

Sketch of the proof:

The orbit space $cc(K)/G$ is a compact AR. Let $\epsilon > 0$ and x_0 the center of symmetry. We construct equivariant maps

$$f_1 : cc(K) \to cc(K) \quad \text{and} \quad f_2 : cc(K) \to cc(K)$$

ϵ-close to the identity map $1_{cc(K)}$ such that

$$im\ f_1 \subset rint\ K \quad \text{and} \quad im\ f_2 \cap rbd\ K \neq \emptyset.$$

Indeed,

$$f_1(A) = x_0 + t(A - x_0), \qquad A \in cc(K), \qquad t \in (1-\epsilon, 1)$$

$$f_2(A) = \{x \in K \mid d(x, A) \leq \epsilon\}, \qquad A \in cc(K).$$

Then f_1 and f_2 induce continuous maps

$$\widetilde{f}_1 : cc(K)/G \to cc(K)/G \quad \textit{and} \quad \widetilde{f}_2 : cc(K)/G \to cc(K)/G$$

also satisfying the properties of Toruńczyk's Theorem. □

Theorem

Let G be a compact group acting affinely on a Keller compactum K. If there is a G-fixed point $x_0 \in \operatorname{rint} K$, then the orbit spaces $2^K/G$ and $cc(K)/G$ are homeomorphic to the Hilbert cube Q.

THE END