Totally periodic graph manifolds

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Definition Let Φ be a flow on a closed 3-manifold M. We say that Φ is a *pseudo-Anosov* flow if the following conditions are satisfied:

- For each $x \in M$, the flow line $t \to \Phi(x,t)$ is C^1, it is not a single point, and the tangent vector bundle $D_t\Phi$ is C^0 in M.
- There are two (possibly) singular transverse foliations Λ^s, Λ^u which are two dimensional, with leaves saturated by the flow and so that Λ^s, Λ^u intersect exactly along the flow lines of Φ.

- There are a finite number (possibly zero) of periodic orbits $\{\gamma_i\}$, called *singular* orbits. A stable/unstable leaf containing a singularity is homeomorphic to $P \times I/f$ where P is a p-prong in the plane and f is a homeomorphism from $P \times \{1\}$ to $P \times \{0\}$. In addition, p is at least 3.
- In a stable leaf all orbits are forward asymptotic, in an unstable leaf all orbits are backward asymptotic.

Definition A pseudo-Anosov flow without singular orbits is an *Anosov* flow.

Manifolds that admit pseudo-Anosov flows

- have $\mathbb{R}^3$ as a universal cover
- have infinite fundamental group with exponential growth
- are irreducible

Definition A *graph manifold* is an irreducible 3-manifold where all of the pieces of the torus decomposition are Seifert.

Definition In relation to a pseudo-Anosov flow, a Seifert fibered piece is *periodic* if the piece admits a Seifert fibration for which a regular fiber is freely homotopic to a closed orbit of the flow.

Definition A graph manifold in which all pieces of the torus decomposition are periodic is *totally periodic*.

Fundamental objective: Classify totally periodic graph manifolds.

Method:

- Show that totally periodic graph manifolds with pseudo-Anosov flow can be described using surfaces called *fat graphs*.
- Study fat graphs.
- Perform Dehn surgery on circle bundles over fat graphs.

Definition A *Birkhoff annulus* is an immersed annulus so that each boundary component is a closed orbit of the flow and the interior of the annulus is transverse to the flow.

Constructing totally periodic graph manifolds

- Start with "building blocks" – solid tori each containing a Birkhoff annulus.
- Glue these together around periodic orbits so that only boundary tori transverse to the flow remain (incoming and outgoing).
- Glue these pieces together incoming boundary torus to outgoing boundary torus.

Definition Given a surface Σ with boundary that retracts onto a graph X, Σ is a *fat graph* for X and X is *flow graph* if:

(i) the valence of every vertex is an even number.

(ii) the set of boundary components of Σ can be partitioned into two subsets so that for every edge e of X, the two sides of e in Σ lie in different subsets of this partition.

Note We do not require Σ to be orientable.

Remark A vertex of valence $2p$ corresponds to a p-prong.

Definition A flow graph is *irreducible* if each vertex has a valence of at least 4.

Definition An irreducible flow graph is a *generating graph* if each of the boundary components of the corresponding surface retracts onto an even number of edges when the surface is retracted onto the graph.

Example (Bonatti, Langevin 1994) The punctured Möbius strip admits a generating graph with 1 vertex.

Theorem 1 (W) *Spheres with 2,3, or 5 boundary components do not admit generating graphs. A torus with 3 boundary components does not admit a generating graph.*

Theorem 2 (W) *All other orientable surfaces of genus* g *with* b *boundary components and* $x \leq b - x$ *incoming boundary components admit a generating graph with* v *vertices if and only if*

- $b \geq 2$,
- $v + b$ *is even,*
- $x \geq 1 - g + (b - v)/2$, *and*
- $v \leq b - 2 + 2g$, *with strict inequality if* v *is odd and* $g = 0$.

More on Seifert fibered spaces:

- Start with a compact surface F of genus g and b boundary components and drill out $n+1$ disks, giving a surface F_0
- Cross F_0 with S^1 to obtain a 3-manifold M_0 with torus boudary components.
- The bundle has a cross-section $s : F_0 \to M_0$.

- Define for each simple closed curve in a component of ∂M_0 a slope $\mathbb{Q} \cup \{\infty\}$, where the section defines slope $\{0\}$ and the fiber defines slope ∞.
- Glue $n+1$ solid tori back onto M_0.
- The glueing of the i-th solid torus identifies the boundary of a meridian disk to some curve a_1(fiber)+b_i(section) in ∂M_0.

Remark Seifert fibered spaces are be obtained by performing Dehn surgery on circle bundles.

Definition The *Seifert invariant* for a Seifert fibered space F is

$$\Sigma(\pm g, b; a_0/b_0, a_1/b_1, ..., a_n/b_n),$$

where $\pm$ is $+$ if F is orientable and $-$ if non-orientable. The rational numbers a_i/b_i are treated as an unordered $(n+1)$-tuple.

Remark The circle bundles over fat graphs are

$$\Sigma(\pm g, b; 0, 0, ..., 0).$$

Surgeries

- We can perform any a_i/b_i Dehn surgery at any of the periodic orbits to obtain a pseudo-Anosov flow.
- Doing a/b surgery on a p-prong (p can be 1 or 2) yields an ap-prong.

Any periodic piece of a totally periodic graph manifold has

$$\Sigma(\pm g, b; 0, a_1/b_1, ..., a_n/b_n,$$
$$c_1/b_{n+1}, c_2/b_{n+2}, ..., c_{2m-1}/b_{n+2m-1}, c_{2m}/b_{n+2m})$$

where $\pm g, b$ corresponds to a fat graph that admits a generating graph with n vertices, and each $c_j > 1$.

Glueing Seifert pieces:

- For each Seifert fibered manifold (the periodic pieces) and each boundary torus T, select a *vertical/horizontal basis* of $H_1(T, \mathbb{Z})$.
- Select a pairing between boundary tori (T, T').
- Choose a two-by-two matrix $M(T, T')$ with integer coefficients that is <u>not</u> upper triangular.

These give all of the totally periodic graph manifolds.

Theorem 3 (W) *A b-punctured sphere that admits a generating graph with v vertices admits a generating graph whose vertices have valence* $\alpha_1, ..., \alpha_v$ *if and only if*

- $\alpha_1 + ... + \alpha_v = 2v + 2b - 4,$ *and*
- *some subset of* $\{\alpha_1, ..., \alpha_v\}$ *sums to* $(\alpha_1 + ... + \alpha_v)/2.$

Theorem 4 (W) *Any orientable surface of positive genus and any non-orientable surface that admits a generating graph with v vertices admits a generating graph whose vertices have valence $\alpha_1, ..., \alpha_v$ if and only if*

$\alpha_1 + ... + \alpha_v = 2v + 2b + 4g - 4$ *or*

$\alpha_1 + ... + \alpha_v = 2v + 2b + 2k - 4$, *respectively.*

Thank you