Groups of uniform homeomorphisms of covering spaces

Tatsuhiko Yagasaki

(Kyoto Institute of Technology)

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§1. Problem

(M, d) : Non-Compact Metric Manifold

$\mathcal{H}^u(M, d)$: Group of Uniform Homeomorphisms of (M, d) (Uniform Topology)

Goal : Understand (Local and Global) Topological Properties of $\mathcal{H}^u(M, d)$

(1) Local and Global Deformation Properties of
Spaces of Uniform embeddings in (M, d)

(2) $\mathcal{H}^u(M, d)$ Local Contractibility & Homotopy Type

(3) Dependence of These Properties on
Behavior of Metric d near Ends of M

(LD) = Local Deformation Property for Uniform embeddings

(GD) = Global Deformation Property for Uniform embeddings

§2. Deformation Property for Uniform embeddings

(M, d) : Metric n-manifold $\quad X, Z \subset M$

$\mathcal{E}^u_*(X, M; Z)$: Space of uniform proper embeddings

$f : X \to M \quad$ s.t. $\quad f = \mathrm{id}$ on $X \cap Z$

(Sup-metric and Uniform Topology)

For any subset A of M we define a condition $(\mathrm{LD})_M$

- The definition is motivated by

Local Deformation Theorem for Top Embeddings

by R. D. Edwards and R. C. Kirby (1971)

Definition. $A \subset M$

$A : (\mathrm{LD})_M \iff$

$\forall\,(X, W', W, Z, Y)$ s.t. $X \subset A, \quad X \subset_u W' \subset W \subset M, \quad Z \subset_u Y \subset M$

$\exists$ Neighborhood $\mathcal{W}$ of $i_W : W \subset M$ in $\mathcal{E}^u_*(W, M; Y)$

$\exists$ Homotopy $\varphi : \mathcal{W} \times [0,1] \longrightarrow \mathcal{E}^u_*(W, M; Z)$ s.t.

(1) $\forall\, f \in \mathcal{W}$

(i) $\varphi_0(f) = f$ (ii) $\varphi_1(f) = \mathrm{id}$ on X

(iii) $\varphi_t(f) = f$ on $W - W'$ and $\varphi_t(f)(W) = f(W)$ $(t \in [0,1])$

(iv) if $f = \mathrm{id}$ on $W \cap \partial M$ then $\varphi_t(f) = \mathrm{id}$ on $W \cap \partial M$ $(t \in [0,1])$

(2) $\varphi_t(i_W) = i_W$ $(t \in [0,1])$

$M : (\mathrm{LD}) \iff M : (\mathrm{LD})_M$

Theorem (R. D. Edwards and R. C. Kirby (1971))

A : Relatively compact $\Longrightarrow A : (\mathrm{LD})_M$

Formal Properties of (LD)

(1) (Invariance under Uniform Homeo's)

$$M \cong N \text{ (uniform homeo)}, \;\; M : \text{(LD)} \implies N : \text{(LD)}$$

(2) (Restriction) $A \subset B \subset M,\; B : \text{(LD)}_M \implies A : \text{(LD)}_M$

(3) (Addition) $A \subset_u U \subset M,\;\; B \subset M$

$$U,\, B : \text{(LD)}_M \implies A \cup B : \text{(LD)}_M$$

(4) (Relatively compact subsets) $A \subset M, \quad K \subset M$: Relatively compact

$$A : \text{(LD)}_M \iff A \cup K : \text{(LD)}_M$$

(5) (Neighborhoods of Ends) $M = K \cup \cup_{i=1}^m L_i$

$K \subset M$: Relatively compact

$L_i \subset M$: Closed, L_i : n-manifold, $d(L_i, L_j) > 0 \;\; (i \neq j)$

$$M : \text{(LD)} \iff L_i : \text{(LD)} \;\; (i = 1, \cdots, m)$$

Examples

[1] Metric covering spaces over compact manifolds

Definition. $\pi : (X, d) \to (Y, \varrho)$: Metric covering projection

$\iff$ $(\natural)_1$ $\exists\, \mathcal{U}$: Open cover of Y s.t. $\forall\, U \in \mathcal{U}$

$\pi^{-1}(U) = \cup_i U_i$: Disjoint union of Open subsets of X

$\pi : U_i \cong U$: Isometry

$(\natural)_2$ $\forall\, y \in Y$ $\pi^{-1}(y)$: uniformly discrete in X

$(\natural)_3$ $\varrho(\pi(x), \pi(x')) \leq d(x, x')$ $(x, x' \in X)$

- This notion is just a metric version of Riemannian covering spaces

Theorem.

$\pi : (M, d) \to (N, \varrho)$: Metric covering projection
N : Compact manifold
$\Longrightarrow$ (M, d) : (LD)

$(\because)$ Edwards - Kirby Deformation Theorem, Arzela - Ascoli theorem

Additivity of $(\mathrm{LD})_M$

[2] Metric manifolds with Local geometric group actions

(M, d) : Metric manifold

Definition. G : Discrete group, $\Phi : G \times M \longrightarrow M$: Continuous action

(1) Φ : locally isometric $\iff$ $\forall\, x \in M \;\; \exists\, \varepsilon > 0$ s.t.

$(\natural)_x$ each $g \in G$ maps $O_\varepsilon(x)$ isometrically onto $O_\varepsilon(gx)$

(2) Φ : locally geometric $\iff$ Φ : proper, cocompact and locally isometric

Theorem.

(M, d) admits Locally geometric group action $\Longrightarrow$ (M, d) : (LD)

$(\because)$ Result on Metric covering projections + Additivity of $(\mathrm{LD})_M$

[3]

Euclidean Space	$\mathbb{R}^n$
Hyperbolic Space	$\mathbb{H}^n$
Cylindrers	$N \times \mathbb{R}$ $(N$: Compact, Product Metric$)$
Ends of these spaces	

Global deformation property for Uniform embeddings

Euclidean ends

(1) Special Feature of Euclidean space : $\exists$ Similarity Transformations

(2) Conjugation with Similarity Transformations

$$\text{(LD)} \implies \text{(GD)} \text{ in Euclidean ends}$$

(3) (GD) in Bi-Lipschitz Euclidean Ends

§4. Deformation Property for Uniform homeomorphisms

(M,d) : Metric Manifold $\quad A \subset M$

$\mathcal{H}^u_A(M,d) =$ Group of uniform homeo's of (M,d) which fix A pointwise

(Uniform Topology)

$\bigcup$ open & closed

$\mathcal{H}^u_A(M,d)_b = \{h \in \mathcal{H}^u_A(M,d) \mid d(h, \mathrm{id}_X) < \infty\}$ (Bounded Uniform Homeo's)

$\bigcup$

$\mathcal{H}^u_A(M,d)_0 =$ Connected component of id_X in $\mathcal{H}^u_A(M,d)$

Local Deformation Property :

(M,d) : (LD) $\implies$ $\mathcal{H}^u(M,d)$: Locally contractible

- Results in Previous section can be applied.

Global Deformation Property :

Definition. Euclidean ends $\mathbb{R}^n_r = \mathbb{R}^n - O_r(\mathbf{0})$ $(r > 0)$

(1) a bi-Lipschitz n-dim Euclidean end of (M, d)

$=$ a closed subset L of M which admits

a bi-Lipschitz homeo $\theta : \mathbb{R}^n_1 \approx (L, d|_L)$ with $\theta(\partial \mathbb{R}^n_1) = \mathrm{Fr}_M L$

∘ $L_r = \theta(\mathbb{R}^n_r)$ $(r \geq 1)$

(2) L : isolated $\iff$ $d(M - L, L_r) \longrightarrow \infty$ as $r \longrightarrow \infty$

(GD) in Bi-Lipschitz Euclidean Ends

Theorem.

(M, d) has Disjoint Isolated bi-Lipschitz Euclidean ends $L(1), \cdots, L(m)$

$L_r := L(1)_r \cup \cdots \cup L(m)_r$ $(r \geq 1)$

$\Longrightarrow$ $\exists$ SDR $\varphi_t : \mathcal{H}^u(M, d)_b \searrow \mathcal{H}^u_{L_3}(M, d)_b$

Examples

(1) $\mathcal{H}^u(\mathbb{R}^n)_b \simeq *$ $\quad$ $((\because)\ \mathcal{H}^u(\mathbb{R}^n)_b \searrow \mathcal{H}_{\mathbb{R}^n_3}(\mathbb{R}^n) \approx \mathcal{H}_\partial(B(3)) \simeq *)$

(i) $\mathcal{H}^u(\mathbb{R})_b \approx \ell_\infty$ (K. Mine, K. Sakai, T. Yagasaki and A. Yamashita)

(ii) **Conjecture.** $\mathcal{H}^u(\mathbb{R}^n)_b \approx \ell_\infty$ $(n \geq 1)$

(2) $n = 2$

N : a compact connected 2-manifold with non-empty boundary

$C_1, \cdots, C_m$: some boundary circles of N $(m \geq 1)$

$C = C_1 \cup \cdots \cup C_m$

$M = N - C$

d : a metric on M s.t.

each ends of M is an isolated bi-Lipschitz Euclidean ends

$\Longrightarrow$ $\mathcal{H}^u(M)_0 \simeq *$ $\quad$ $((\because)\ \mathcal{H}^u(M)_0 \searrow \mathcal{H}_{L_3}(M)_0 \approx \mathcal{H}_C(N)_0 \simeq *)$

End of Talk

Thank you very much !