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The title refers to a paper which has already appeared, [F]. That paper was the first that I wrote, rereading it now I agree with the response that it is unreadable. So there is a need for a less rigorous and more intuitive version.

Following Kunen's suggestion in [T], I was trying to prove that normal Moore spaces are collectionwise normal, assuming Gödel's axiom of constructibility. What I did prove was

Theorem. (V = L). *Normal Hausdorff spaces of character* $\leq c$ *are collectionwise Hausdorff.*

Definition. A space X has *character* $\leq c$ if every $y \in X$ has a neighborhood base, $\{B(y,\gamma): \gamma < c\}$.

Definition. A subset Y of a space X is *closed discrete* if every point of X has a neighborhood intersecting at most one point of Y.

Definition. A closed discrete set Y can be *separated* if there is a family of disjoint open sets $\mathcal{U} = \{V_y : y \in Y\}$ with $y \in V_y$.

Definition. A space X is (κ-) *collectionwise Hausdorff* if every closed discrete set (of cardinality $\leq\kappa$) can be separated.

Let me first remark that this theorem extends

Tall's Theorem. [T]. *In a certain model of set theory, normal Hausdorff spaces of character* $\leq c$ *are* κ*-collectionwise Hausdorff for all* $\kappa < \aleph_{\omega_1}$.

Secondly, there are limitations on improving this theorem. Some additional set theoretic assumption is necessary because assuming Martin's Axiom plus the negation of the continuum hypothesis, there is a separable, normal, not collectionwise Hausdorff Moore space, [T]. Bing's example G [B] is a normal not collectionwise Hausdorff space, so the character restriction is necessary. Finally, while there still is hope of proving the normal Moore space conjecture in L, such a proof cannot extend to prove that normal spaces of character $\leq c$ are collectionwise normal [F'].

Before proceeding to the proof, let us define some notions about cardinals.

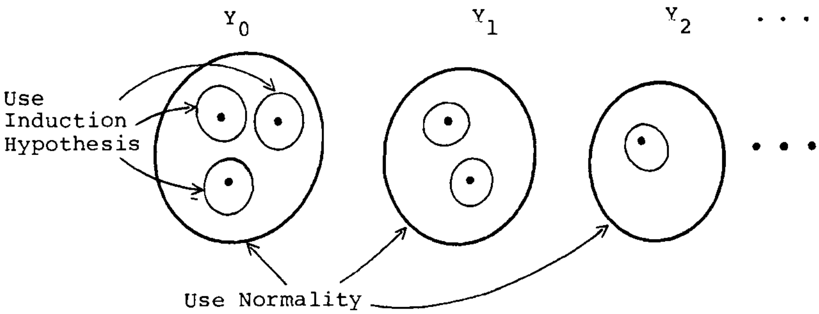
Definition. A cardinal κ is regular if the union of $<\kappa$ sets of cardinality $<\kappa$ has cardinality $<\kappa$. A cardinal is singular otherwise. For example, $\aleph_1$ is regular because a countable union of countable sets is countable. Another example: $\aleph_\omega$ is singular because the cardinality of the union of sets of cardinality $\aleph_n$, $n \in \omega$, has cardinality $= \aleph_\omega$. If κ is shown singular by a countable union we say that κ has cofinality ω.

Proof of Theorem. We prove normal Hausdorff spaces of character $\leq c$ are κ-collectionwise Hausdorff by induction on κ.

Case I. κ is finite. Use Hausdorff.

Case II. $\kappa = \omega$. Use regularity.

Case III. κ is singular of cofinality ω. Let Y be closed discrete set of cardinality κ. Then Y can be written $Y = \bigcup_{i \in \omega} Y_i$ where the cardinality of $Y_i < \kappa$. Y can be separated as indicated below.



Case IV. κ is regular. Instead of jumping in, let's do a couple similar but simpler proofs to develop a rhythm.

Theorem. (Bernstein). *There is a subset* Z *of* $\mathbb{R}$ *such that neither* Z *nor* $\mathbb{R}$ - Z *contains a perfect closed subset of* $\mathbb{R}$.

Proof. Well order the set of perfect closed subsets of $\mathbb{R}$ and inductively assign points to Z or $\mathbb{R}$ - Z. The details are left to the reader. We take note of three things:

1. There are c steps--$\mathbb{R}$ has c points.
2. There are c tasks--$\mathbb{R}$ has c perfect closed subsets.
3. Each task can be done after <c steps--every perfect closed set has c points.

The next example is too contrived to be called a theorem, so call it

Exercise. Suppose X *is a regular Hausdorff space in which two disjoint closed sets, one of which is countable, can be separated;* $Y = \{y_\gamma : \gamma < \omega_1\}$ *is a closed discrete collection of points; and* $\mathfrak{F}$ *is a family of* $\aleph_1$ *open covers* $\mathfrak{U} = \{U_\gamma : \gamma < \omega_1\}$ *of* Y *such that* $y_\gamma \in U_\gamma$.

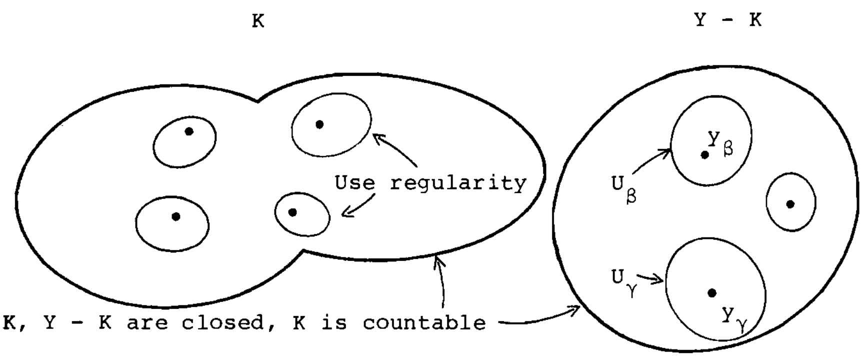
Then either Y *can be separated or there is an* $H \subset Y$ *that can not be separated from* Y - H *by any* $\mathfrak{U} \in \mathfrak{F}$*; i.e. for all* $\mathfrak{U} \in \mathfrak{F}$,

$$\bigcup_{y_\gamma \in H} U_\gamma \cap \bigcup_{y_\gamma \notin H} U_\gamma \neq \emptyset$$

Proof. Let us attempt to define such an H by the above method, and see what conditions we need.

1. There are ω_1 steps--Y has ω_1 points.
2. There are ω_1 tasks-- $\mathcal{F}$ has ω_1 $\mathcal{U}$'s.
3. Each task can be done after $<\omega_1$ steps.

To satisfy condition 3 we need to know that for any countable subset K of Y, and any $\mathcal{U} \in \mathcal{F}$, there are y_β, $y_\gamma \notin K$ so that $U_\beta \cap U_\gamma \neq \emptyset$. We do task U by assigning y_β to H and y_γ to Y - H. If for some K and $\mathcal{U}$ there are no such y_β, y_γ, then Y can be separated, as shown below.



Of course, the idea of the above exercise is that if $\mathcal{F}$ were all covers $\mathcal{U}$ of Y and Y cannot be separated then X is not normal.

Let us return to the proof of the theorem. For concreteness, let us consider the case $\kappa = \omega_1$. (The proof for arbitrary regular κ is the same.) Suppose that X has character ω_1, and that $Y \subset X$ is closed, discrete, and cannot be separated. To show that X is not normal, it is sufficient to do all the tasks $\mathcal{U}_f$, where f is a function from ω_1 to ω_1, and $U_\gamma = B(y_\gamma, f(\gamma))$. The problem is that there are 2^{ω_1} tasks and only ω_1 steps.

Let us note that the $\mathcal{U}_f$'s do not have to be done separately. When we assign y_β to H and y_γ to Y - H, we do all tasks $\mathcal{U}_f$ where

$B(y_\beta, f(\beta)) \cap B(y_\gamma, f(\gamma)) \neq \emptyset$.

When we are faced with doing tasks indexed by functions from ω_1 to ω_1 in ω_1 steps, it is often helpful to use

The Technique of $\Diamond$. Suppose a family of tasks is indexed by the functions from ω_1 to ω_1. Suppose almost all initial segment tasks can be done. Then, assuming $\Diamond$, in ω_1 steps we can do all the tasks.

Definition. $\Diamond$ is the assertion of the existence of a sequence $\{\Gamma_\gamma : \gamma < \omega_1\}$ of functions from γ to ω_1 such that for every f from ω_1 to ω_1 and every C closed unbounded in ω_1, there is $\gamma \in C$, $f|\gamma = \Gamma_\gamma$.

Definition. "Almost all initial segments" means that for every function f from ω_1 to ω_1, there is a closed unbounded subset C_f of ω_1 so that for all $\gamma \in C_f$, the task corresponding to $f|\gamma$ (the restriction of f to γ) can be done.

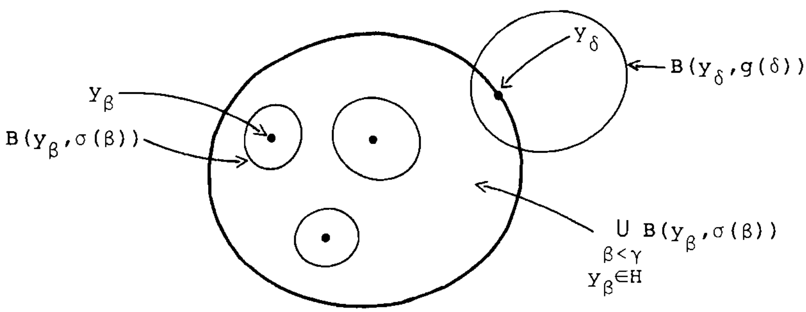
Definition. Let σ be a function from a countable ordinal γ to ω_1. T_σ, the task corresponding to σ, can be done, if, after the first γ steps have been done, the γ^{th} step can be done in a way that does the task indexed by g for all g extending σ.

(Because we want to give the reader an intuitive idea of how to use $\Diamond$ in a variety of cases, the last definition is somewhat vague. Here we give an expansion and clarification that should be omitted on first reading.

In the application in this paper, we can tell whether T_σ can be done at the γ^{th} step no matter how the first γ steps have been done. Somewhat more involved is the construction of a Souslin tree from $\Diamond$, where whether T_σ can be done depends on the construction. This prevents us from explicitly defining C_f from f, but does not prevent from proving that no matter what the construction, there is some closed unbounded C_f.)

In the exercise, we did a task T by assigning two new points y_α, y_β with $U_\alpha \cap U_\beta \neq \emptyset$. This method does not suffice to do a task T_σ because σ does not assign neighborhoods to new points. In fact, because the space is Hausdorff, if y_α, y_β are two points not assigned neighborhoods by σ, there is g extending σ with $B(y_\alpha, g(\alpha)) \cap B(y_\beta, y(\beta)) = \emptyset$. So to do T_σ we must assign a new point so that all its neighborhoods meet the neighborhoods of points already assigned.

Call $\overline{\bigcup_{\beta<\gamma, y_\beta \in H} B(y_\beta, \sigma(\beta))} \cap \{y_\delta : \delta \geq \gamma\}$ the H-limit points of σ. Similarly define the (y-H)-limit points of σ. The way to do task T_σ is to assign an H-limit point y_δ to Y-H (or a (Y-H)-limit point to H).



No matter how we extend σ to g, task T_g is done.

Proof of Technique of $\Diamond$. Let $\{\Gamma_\gamma : \gamma < \omega_1\}$ be the sequence $\Diamond$ says exists. At step γ, do task T_{Γ_γ}, if possible. By $\Diamond$, for every f there is $\gamma \in C_f$ with $f|\gamma = \Gamma_\gamma$. So every f is done at some initial segment.

(We have glossed over a technical problem. In order to do task T_{Γ_γ}, we may have to assign y_δ, with $\delta > \gamma$. Then we simply assign the points y_η, $\gamma \leq \eta < \delta$ arbitrarily. Because ω_1 is regular, the set of η such that no y_δ, $\delta \geq \eta$, is already assigned contains a closed unbounded set C. Now $C \cap C_f$ is again closed, unbounded.)

Although $\Diamond$ does a variety of wonderful things, to prove our theorem we need

Technique of $\Diamond_F$. Suppose a family of tasks is indexed by the functions from ω_1 to ω_1. If many initial segment tasks can be done, then, assuming V = L, in ω_1 steps we can do all the tasks.

Definition. "Many"--for all f there is a set A_f, which meets every closed unbounded subset of ω_1, for which $\gamma \in A_f$ implies that task $T_{f|\gamma}$ can be done.

Now, back to proving the theorem by induction. If many initial segments have limit points, X is not normal. (Assuming V = L and using the technique of $\Diamond_F$.) So we assume not and prove that Y can be separated. Explicitly, "assume not" means that there is a function f from ω_1 to ω_1 and a closed unbounded set C such that for all $\gamma \in C$, $f|\gamma$ has no limit points.

Then, for $\gamma \in C$

$$V_\gamma = X - \overline{\bigcup_{\beta<\gamma} B(y_\beta, f(\beta))}$$

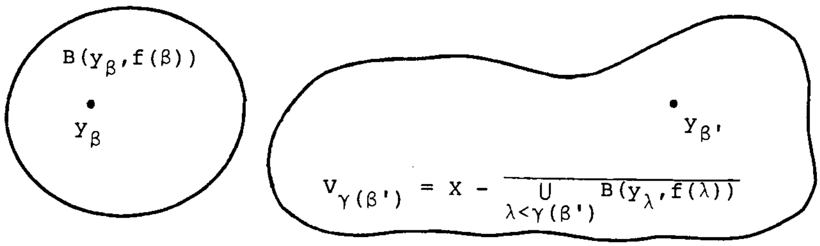
is an open set containing $\{y_\delta : \delta \geq \gamma\}$. Because C is closed, $\gamma(\beta) = \sup\{\gamma \in C : \gamma < \beta\}$ is in C. Because C is unbounded, only countably many β's have the same $\gamma(\beta)$. So there are open sets W_β so that if $\beta < \beta'$, $\gamma(\beta) = \gamma(\beta')$, then $W_\beta \cap W_{\beta'} = \emptyset$.

Now Y can be separated, because $\beta < \beta'$ implies

$(B(y_\beta, f(\beta)) \cap V_{\gamma(\beta)} \cap W_\beta \cap (B(y_{\beta'}, f(\beta')) \cap V_{\gamma(\beta')} \cap W_{\beta'} = \emptyset$.

Case 1. $\gamma(\beta') \leq \beta < \beta'$. Then $\gamma(\beta) = \gamma(\beta')$.

Case 2. $\beta < \gamma(\beta') \leq \beta'$.



Case V. $\kappa > \text{cf } \kappa > \omega$. The basic idea is quite similar to Case IV. We can inductively define an H showing that X is not normal if we can find a sequence of initial segment tasks doing all the tasks. If there is no such sequence, we can use that fact to separate Y.

H can be defined if there is an order such that every function has at least one initial segment with many limit points.

Rather than do Case V in detail, we simply list the differences between Case IV and Case V.

1. We consider all possible orders on Y.
2. We consider only cf κ many initial segments of a function.
3. In the condition when H can be defined "many" goes with limit points rather than initial segments.
4. The steps are arranged into supersteps. A superstep is an induction to ruin all initial segments of a given length.
5. If H cannot be defined, we don't separate Y in one blow. We separate as many points as we can; then reorder so that every bad point has a lower index. Because there are no infinite descending sequences of ordinals, after ω steps, there are no bad points left. We then separate our countably many separations as in Case III.

The gory details of when H can be defined finish this article. Let C be a set of cardinals of order type cf κ cofinal in κ. H can be defined if there is a permutation π of κ such that for all functions f from κ to ω_1 there is $\gamma \in C$ such that

$$\text{card } [\overline{\bigcup_{\pi(\beta)<\gamma} B(y_\beta, f(\pi(\beta)))} \cap \{y_\delta : \pi(\delta) \geq \gamma\}] > \gamma.$$

References

[B] R. H. Bing, *Metrization of topological spaces*, Canad. J. Math. 3 (1951), 175-186.

[F] W. G. Fleissner, *Normal Moore spaces in the constructible universe*, Proc. Amer. Math. Soc. 46 (1974), 294-298.

[F'] ________, *A normal, collectionwise Hausdorff, not collectionwise normal space*, Gen. Topol. and Appl. 6 (1976), 57-64.

[T] F. D. Tall, *Set theoretic consistency results and topological theorems*, University of Wisconsin Thesis, 1969.

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