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The following result has long been known to Russians and is considered elementary, but the proof does not seem to have appeared in print:

Theorem 1. Let X *be a Hausdorff space and let* $f: X \to Y$ *be a perfect light map. If* Y *is regular, then* ind $X \leq$ ind Y.

(A continuous function f is *perfect* if it is closed and $f^{-1}(y)$ is compact for all $y \in Y$. It is *light* if $f^{-1}(y)$ is totally disconnected for all $y \in Y$.)

The proof makes use of the following trivial lemma:

Lemma 2. Let G_1 *and* G_2 *be disjoint open subsets of a space* X *and let* K *be a set whose closure is contained in* $G_1 \cup G_2$. *Then* $Bd(K \cap G_1) = Bd\, K \cap G_1$. *In particular, if* K *is clopen, so is* $K \cap G_1$.

Proof of Theorem 1. Let x be a point of X and let $F = f^{-1}(f(x))$. Let U be an open neighborhood of x. By zero-dimensionality of F, there exist disjoint closed sets F_1 and F_2 such that $x \in F_1 \subset U_1$, $F_1 \cup F_2 = F$. Let V_1 and V_2 be disjoint open subsets of X containing F_1 and F_2 respectively. Let $G_1 = V_1 \cap U$, $G_2 = V_2$.

Let $V = G_1 \cup G_2$. Because f is a closed map, $[f(V^c)]^c$ is an open set containing f(x) whose inverse image is contained in V.

The rest of the proof goes by induction. Suppose ind Y = 0. Then there exists a clopen set K containing f(x) and contained in $[f(V^c)]^c$. The inverse image of K is a clopen set contained in V; hence by the lemma, $f^{-1}(K) \cap G_1$ is clopen, and we have $x \in f^{-1}(K) \cap G_1 \subset G_1 \subset U$.

Suppose the theorem has been proven for ind $Y \leq n$, and let ind $Y = n+1$. By regularity of Y, there exists a neighborhood A of f(x) whose closure is contained in $[f(V^c)]^c$ and whose boundary is of ind $\leq n$. Since Bd $f^{-1}(A) \subset f^{-1}(\text{Bd } A)$ by continuity it follows that Bd $f^{-1}(A)$ has small inductive dimension $\leq n$ by the induction hypothesis. By the lemma, Bd $f^{-1}(A) \cap G_1 =$ Bd $(f^{-1}(A) \cap G_1)$, so that $f^{-1}(A) \cap G_1$ is a neighborhood of x contained in G_1 (hence in U) whose boundary has small inductive dimension $\leq n$, as was to be shown.

The only place in the above proof where "perfect" was used was in getting disjoint closed (and relative open) subsets of $f^{-1}(y)$ into disjoint open subsets of X. This can be done in a number of alternative ways. For example (we take "regular" and "normal" to include "Hausdorff"):

Theorem 2. Let X *be a regular space and let* $f: X \to Y$ *be a closed map such that* $f^{-1}(y)$ *is Lindelöf (or locally compact) and zero-dimensional for all* $y \in Y$. *If* Y *is regular, then* ind $X \leq$ ind Y.

Theorem 3. Let X *be a normal space and let* $f: X \to Y$ *be a closed map such that* $f^{-1}(y)$ *is zero-dimensional for all* $y \in Y$. *Then* ind $X \leq$ ind Y.

More generally, we have:

Theorem 4. Let X *be a topological space and let* $f: X \to Y$ *be a closed map such that* $f^{-1}(y)$ *is* C*-*embedded and zero-dimensional for all* $y \in Y$. *If* Y *is regular, then* ind $X \leq$ ind Y.

The following examples show the necessity of "Hausdorff" in Theorem 1 and "normal" in Theorem 3.

Example 5. Let X be the space consisting of a sequence of closed and isolated points x_n which converge to two distinct closed points, x and z. Let Y be the space obtained by identifying x and z, and let f be the resulting map. (Clearly, Y is homeomorphic to $\omega+1$.) Then f is a perfect light map, and ind Y = 0, but ind X = 1.

Example 6. Let Z be a version of Ψ [2, Exercise 5I] which is zero-dimensional but not strongly zero-dimensional [3]. Let $g: Z \to [0,1]$ be a continuous function such that $g^{-1}(0)$ and $g^{-1}(1)$ are not contained in disjoint clopen sets. Let X be the space which is gotten by identifying $g^{-1}(1)$ to a single point and letting the neighborhoods of this point have a base consisting of the sets $g^{-1}(1-\varepsilon,1]$. Let the rest of X be given the relative topology as a subspace of Z. Then X is Tychonoff, and ind X = 1.

Let $f: X \to Y$ be the map resulting from identifying all nonisolated points of X to a single point, Y the resulting space (which is homeomorphic to $\omega+1$). Then f is closed, and $f^{-1}(y)$ is closed and zero-dimensional for all $y \in Y$. But ind Y = 0.

An interesting consequence of Theorem 1 is that the inverse preservation of a class of zero-dimensional spaces under perfect light maps with Hausdorff domain, is equivalent to its inverse preservation under perfect maps with zero-dimensional Hausdorff domain.

Definition 7. Let $\mathfrak{A}$ be a category of topological spaces and let $\mathfrak{B}$ be a full and replete subcategory of $\mathfrak{A}$. Then $\mathfrak{B}$ is [*lightly*] *left-fitting in* $\mathfrak{A}$ if whenever $f: X \to Y$ is a perfect [light] map with $X \in \mathfrak{A}$ and $Y \in \mathfrak{B}$, then $X \in \mathfrak{B}$.

Theorem 8. Let $\mathfrak{B}$ be a category of zero-dimensional Hausdorff spaces. The following are equivalent.

(1) $\mathfrak{B}$ is lightly left-fitting in the category of Hausdorff

spaces.

(2) $\mathcal{B}$ is left-fitting in the category of zero-dimensional Hausdorff spaces.

(3) $\mathcal{B}$ is closed hereditary, and every product of a space in $\mathcal{B}$ with a zero-dimensional compact Hausdorff space is in $\mathcal{B}$.

Proof. That (1) is equivalent to (2) is immediate from Theorem 1. It is clear that (2) implies (3). To prove that (3) implies (2), one adapts the argument in [1], substituting "zero-dimensional" for "Tychonoff" and ζX for βX.

Example 9. The category of N-compact spaces is lightly left-fitting in the category of Hausdorff spaces. (A space is N-*compact* if it can be embedded as a closed subspace in a product of countable discrete spaces.) This follows from Theorem 8, since (3) is clearly satisfied.

Problem 10. Let X be a Hausdorff space and let $f: X \to Y$ be a perfect map such that ind $f^{-1}(y) \leq n$ for all $y \in Y$. Is it true that ind $X \leq$ ind $Y+n$?

This is the natural generalization of Theorem 1, but the proof of Theorem 1 leans so heavily upon the zero-dimensionality of $f^{-1}(y)$ that there seems little hope of an affirmative answer here, even if we assume X and Y to be hereditarily normal.

References

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