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by

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INVERSE PRESERVATION OF SMALL INDUCTIVE DIMENSION

Peter J. Nyikos

The following result has long been known to Russians and is considered elementary, but the proof does not seem to have appeared in print:

Theorem 1. Let X *be a Hausdorff space and let* $f: X \to Y$ *be a perfect light map. If* Y *is regular, then* $\operatorname{ind} X \leq \operatorname{ind} Y$.

(A continuous function f is *perfect* if it is closed and $f^{-1}(y)$ is compact for all $y \in Y$. It is *light* if $f^{-1}(y)$ is totally disconnected for all $y \in Y$.)

The proof makes use of the following trivial lemma:

Lemma 2. Let G_1 *and* G_2 *be disjoint open subsets of a space* X *and let* K *be a set whose closure is contained in* $G_1 \cup G_2$. *Then* $\operatorname{Bd}(K \cap G_1) = \operatorname{Bd} K \cap G_1$. *In particular, if* K *is clopen, so is* $K \cap G_1$.

Proof of Theorem 1. Let x be a point of X and let $F = f^{-1}(f(x))$. Let U be an open neighborhood of x. By zero-dimensionality of F, there exist disjoint closed sets F_1 and F_2 such that $x \in F_1 \subset U_1$, $F_1 \cup F_2 = F$. Let V_1 and V_2 be disjoint open subsets of X containing F_1 and F_2 respectively. Let $G_1 = V_1 \cap U$, $G_2 = V_2$.

Let $V = G_1 \cup G_2$. Because f is a closed map, $[f(V^c)]^c$ is an open set containing f(x) whose inverse image is contained in V.

The rest of the proof goes by induction. Suppose $\operatorname{ind} Y = 0$. Then there exists a clopen set K containing f(x) and contained in $[f(V^c)]^c$. The inverse image of K is a clopen set contained

in V; hence by the lemma, $f^{-1}(K) \cap G_1$ is clopen, and we have $x \in f^{-1}(K) \cap G_1 \subset G_1 \subset U$.

Suppose the theorem has been proven for ind $Y \leq n$, and let ind $Y = n+1$. By regularity of Y, there exists a neighborhood A of f(x) whose closure is contained in $[f(V^c)]^c$ and whose boundary is of ind $\leq n$. Since Bd $f^{-1}(A) \subset f^{-1}(\text{Bd } A)$ by continuity it follows that Bd $f^{-1}(A)$ has small inductive dimension $\leq n$ by the induction hypothesis. By the lemma, Bd $f^{-1}(A) \cap G_1 =$ Bd $(f^{-1}(A) \cap G_1)$, so that $f^{-1}(A) \cap G_1$ is a neighborhood of x contained in G_1 (hence in U) whose boundary has small inductive dimension $\leq n$, as was to be shown.

The only place in the above proof where "perfect" was used was in getting disjoint closed (and relative open) subsets of $f^{-1}(y)$ into disjoint open subsets of X. This can be done in a number of alternative ways. For example (we take "regular" and "normal" to include "Hausdorff"):

Theorem 2. Let X *be a regular space and let* f: X → Y *be a closed map such that* $f^{-1}(y)$ *is Lindelöf (or locally compact) and zero-dimensional for all* $y \in Y$. *If* Y *is regular, then* ind $X \leq$ ind Y.

Theorem 3. Let X *be a normal space and let* f: X → Y *be a closed map such that* $f^{-1}(y)$ *is zero-dimensional for all* $y \in Y$. *Then* ind $X \leq$ ind Y.

More generally, we have:

Theorem 4. Let X *be a topological space and let* f: X → Y *be a closed map such that* $f^{-1}(y)$ *is* C*-*embedded and zero-dimensional for all* $y \in Y$. *If* Y *is regular, then* ind $X \leq$ ind Y.

The following examples show the necessity of "Hausdorff" in Theorem 1 and "normal" in Theorem 3.

Example 5. Let X be the space consisting of a sequence of closed and isolated points x_n which converge to two distinct closed points, x and z. Let Y be the space obtained by identifying x and z, and let f be the resulting map. (Clearly, Y is homeomorphic to $\omega+1$.) Then f is a perfect light map, and ind Y = 0, but ind X = 1.

Example 6. Let Z be a version of Ψ [2, Exercise 5I] which is zero-dimensional but not strongly zero-dimensional [3]. Let $g: Z \to [0,1]$ be a continuous function such that $g^{-1}(0)$ and $g^{-1}(1)$ are not contained in disjoint clopen sets. Let X be the space which is gotten by identifying $g^{-1}(1)$ to a single point and letting the neighborhoods of this point have a base consisting of the sets $g^{-1}(1-\varepsilon,1]$. Let the rest of X be given the relative topology as a subspace of Z. Then X is Tychonoff, and ind X = 1.

Let $f: X \to Y$ be the map resulting from identifying all nonisolated points of X to a single point, Y the resulting space (which is homeomorphic to $\omega+1$). Then f is closed, and $f^{-1}(y)$ is closed and zero-dimensional for all $y \in Y$. But ind Y = 0.

An interesting consequence of Theorem 1 is that the inverse preservation of a class of zero-dimensional spaces under perfect light maps with Hausdorff domain, is equivalent to its inverse preservation under perfect maps with zero-dimensional Hausdorff domain.

Definition 7. Let $\mathcal{A}$ be a category of topological spaces and let $\mathcal{B}$ be a full and replete subcategory of $\mathcal{A}$. Then $\mathcal{B}$ is *[lightly] left-fitting in* $\mathcal{A}$ if whenever $f: X \to Y$ is a perfect [light] map with $X \in \mathcal{A}$ and $Y \in \mathcal{B}$, then $X \in \mathcal{B}$.

Theorem 8. Let $\mathcal{B}$ be a category of zero-dimensional Hausdorff spaces. The following are equivalent.

(1) $\mathcal{B}$ is lightly left-fitting in the category of Hausdorff

spaces.

(2) $\mathfrak{B}$ *is left-fitting in the category of zero-dimensional Hausdorff spaces.*

(3) $\mathfrak{B}$ *is closed hereditary, and every product of a space in* $\mathfrak{B}$ *with a zero-dimensional compact Hausdorff space is in* $\mathfrak{B}$.

Proof. That (1) is equivalent to (2) is immediate from Theorem 1. It is clear that (2) implies (3). To prove that (3) implies (2), one adapts the argument in [1], substituting "zero-dimensional" for "Tychonoff" and ζX for βX.

Example 9. The category of N-compact spaces is lightly left-fitting in the category of Hausdorff spaces. (A space is N-*compact* if it can be embedded as a closed subspace in a product of countable discrete spaces.) This follows from Theorem 8, since (3) is clearly satisfied.

Problem 10. Let X be a Hausdorff space and let $f: X \to Y$ be a perfect map such that $\operatorname{ind} f^{-1}(y) \leq n$ for all $y \in Y$. Is it true that $\operatorname{ind} X \leq \operatorname{ind} Y + n$?

This is the natural generalization of Theorem 1, but the proof of Theorem 1 leans so heavily upon the zero-dimensionality of $f^{-1}(y)$ that there seems little hope of an affirmative answer here, even if we assume X and Y to be hereditarily normal.

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