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by

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In a discussion following R. H. Bing's talk on "The elusive fixed point property" [2] at the 1968 Houston Topology Conference it was observed that the easy proof of Brouwer's Fixed Point Theorem for an interval, obtained by considering points which move right and those moving left, extends to two dimensions with the aid of a Phragmen-Brouwer property but no further. The author tried for some time to find the proper extension of this property with little success. Upon reading a new solution of this two-dimensional problem (and others) by Albrecht Dold [3], the proper extension was eventually found to be a variation of what is sometimes called Alexander's Addition Theorem. Since it is really a restatement of the latter, it is not a new result but is a new way of interpreting and relating known results.

This new approach permits very short simple proofs, using only a few simple concepts (finite Euclidean complexes, mappings, compactness and connectedness properties), of a large number of powerful theorems in Euclidean topology. It most nearly reflects the work of the above-mentioned work of Dold and Chapter Five of M. H. A. Newman's book [5]. Unlike these, it does not explicitly involve any homotopy or homology theory but they are just under the surface (especially the singular theory) so that it could serve as excellent motivation for the introduction of algebraic topology.

The presentation is simplified if we agree to reserve some symbols to represent certain standard objects. Let E^n be Euclidean n-space, D^n its unit disk (all x in E^n with $\|x\| \leq 1$)

and S^n the unit n-sphere (all x in E^{n+1} with $\|x\| = 1$). Homeomorphs of D^n and S^n will be called *n-disks* and *n-spheres*, respectively. An *n-simplex*, σ^n, is the convex hull of n+1 independent points of E^n, called *vertices* of σ^n, and the convex hull of any subset of vertices is called a *face* of σ^n. Let K^n denote an *n-complex*, a finite union of n-simplexes which are pairwise disjoint or meet in a common face (K^n is properly a pair consisting of a collection of n-simplexes and their union). The (n-1)-simplexes which are faces of an odd number of n-simplexes of K^n form an (n-1)-complex ∂K^n called the (mod 2) *boundary* of K^n. An n-complex with empty boundary is called an *n-boundary* and denoted B^n. A map (continuous function) $f:B^n \to X$ is a *boundary map* (in A) if it extends to a map $F:K^{n+1} \to X$ where $\partial K^{n+1} = B^n$ (and $F(K^{n+1}) \subset A$). Dimensional superscripts may be omitted if no confusion results.

The only result needed to establish Alexander's Addition Theorem is the familiar fact, proved below, that the boundary of an n-complex is an (n-1)-boundary. The converse of this is also true (justifying the terminilogy) since the cone over an (n-1)-boundary B is easily seen to be an n-complex with boundary B. Two more comments will make the conditions in the Addition Theorem seem more natural or attainable. First, if two complexes K_1 and K_2 have the same boundary B, for most purposes it is no loss of generality to assume $K_1 \cap K_2 = B$. One need only embed the (finite number of) vertices of B, K_1-B and K_2-B independently in some E^n in order to obtain isomorphic (isometric if need be) copies of B, K_1 and K_2 with the desired properties. Finally, any map of an n-boundary B into a convex set A is a boundary map in A since it extends to the cone over B (linear on each element of the cone).

Lemma. $\partial\partial K^n = \phi$.

Proof. If σ^{n-2} in ∂K^n is a face of m (n-1)-simplexes $\sigma_1,\ldots,\sigma_m$ of ∂K^n, we need only show m is even. Let $\sigma_1',\ldots,\sigma_{m'}'$ be all other (n-1)-simplexes of K^n with face σ^{n-2}. By definition of ∂K^n, σ_i is a face of $2n_i+1$ n-simplexes of K^n and σ_i' is a face of $2n_i'$ such simplexes. This counts each n-simplex with face σ^{n-2} twice (once for each vertex not in σ^{n-2}, see Figure 1) so $\sum_{i=1}^{m}(2n_i+1)+\sum_{i=1}^{m'}2n_i' = 2(\sum_{i=1}^{m}n_i+\sum_{i=1}^{m'}n_i')+m$ is even which implies m is even.

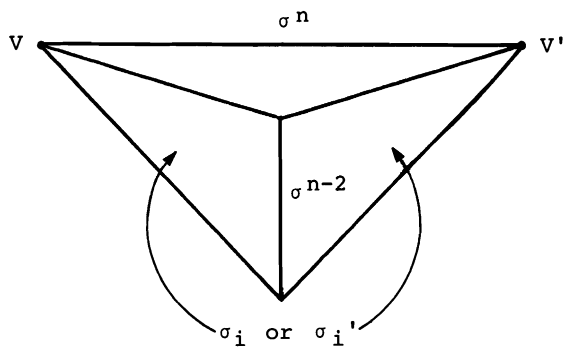

Figure 1

Theorem (Alexander Addition). *For $i = 1,2$ suppose* C_i *is closed in* X, $B^{n-1} = \partial K_i^n = K_1^n \cap K_2^n$ *and* $K_1^n \cup K_2^n = \partial K^{n+1}$. *If* $f:B^{n-1} \to X-C_1 \cup C_2$ *extends to* $f_i:K_i^n \to X-C_i$ *and* $f_1 \cup f_2$ *extends to* $F:K^{n+1} \to X-C_1 \cap C_2$, *then for some* $K^n \subset K^{n+1}$, $\partial K^n = B^{n-1}$ *and* $F(K^n) \subset X-C_1 \cup C_2$.

Proof. Since $F^{-1}(C_i)$ is compact and $F^{-1}(C_1) \cap (F^{-1}(C_2) \cup K_1^n) = (F^{-1}(C_1) \cap F^{-1}(C_2)) \cup (F^{-1}(C_1) \cap K_1^n) = F^{-1}(C_1 \cap C_2) \cup f_1^{-1}(C_1) = \phi$, then K^{n+1} can be subdivided so that the complex K of (n+1)-simplexes meeting $F^{-1}(C_2) \cup K_1^n$ misses $F^{-1}(C_1)$. Then $\partial K = K_1^n \cup K^n$ where K^n misses $F^{-1}(C_1) \cup F^{-1}(C_2)$, i.e. $F(K^n) \subset X-C_1 \cup C_2$($K_1^n$ and K^n have no common n-simplex, see Figure 2). The Lemma gives $\partial(K_1^n \cup K^n) = \partial\partial K = \phi$ so if σ^{n-1} is a face of m n-simplexes of K^n and m' n-simplexes of K_1^n, then m+m' is even. Thus m is odd iff m' is odd and $\partial K^n = \partial K_1^n = B^{n-1}$.

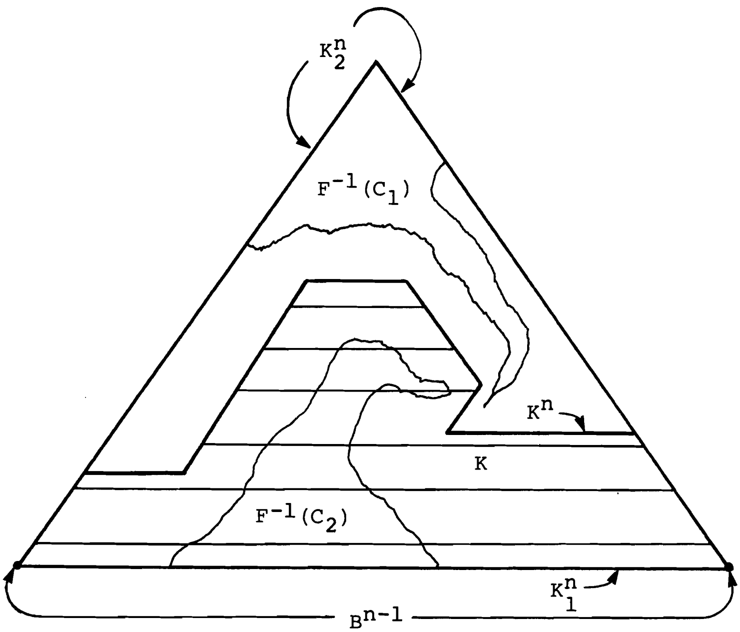

Figure 2

Corollary (No Retraction). *No complex retracts to an n-sphere which is its (mod 2) boundary.*

Proof (by induction). If ∂K^1 is a 0-sphere, K^1 contains an arc joining the two points of ∂K^1 which, being connected, cannot be retracted to the disconnected set ∂K^1. Suppose $n > 0$ and the corollary is true for values less than n. If F retracts $X = K^{n+1}$ to $S = \partial K^{n+1}$, an n-sphere, let C_1 be all of S except for the interior of an n-simplex and C_2 a single point in $S-C_1$. Subdivide K^{n+1} so there is an n-simplex in $S-C_1$ with C_2 in its interior, let K_1^n be this inner n-simplex and K_2^n the union of all other n-simplexes of ∂K^{n+1} (see Figure 3). If f and f_i are the identity on $\partial K_1^n = \partial K_2^n$ and K_i^n respectively, then Alexander's Addition gives a complex K^n with $\partial K^n = \partial K_1^n$ (an (n-1)-sphere) and $F(K^n) \subset X-C_1 \cup C_2$. But then F followed by projection from C_2 retracts K^n to ∂K^n contrary to the induction assumption.

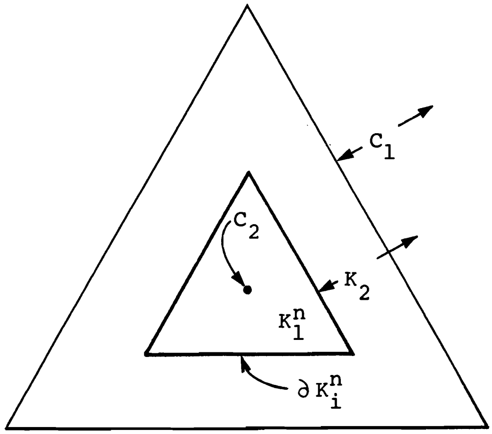

Figure 3

Corollary (Intermediate Value). *If a map* $f:D^n \to E^n$ *is the identity on* S^{n-1} *then* $D^n \subset f(D^n)$, i.e. f *takes on every "value" in* D^n.

Proof. If p is a point of D^n not in $f(D^n)$, let K^n be the cube bounded by the planes $x_i = \pm 1$ which circumscribes D^n and r the radial homeomorphism shrinking K^n to D^n. Then for followed by projection on the (n-1)-sphere ∂K^n from p is a retraction contradicting the previous Corollary.

Corollary (Brouwer Fixed Point). *Every map* $g:D^n \to E^n$ *either leaves a point of* D^n *fixed or stretches a point of* S^{n-1}. *That is, for some* $p \in D^n$, $g(p) = \lambda p$, $\lambda \geq 1$ *and* $p \in S^{n-1}$ *if* $\lambda > 1$.

Proof. Let $f(x) = 2x - g(2x)$ if $\|x\| \leq 1/2$ and $f(x) = x/\|x\| - 2(1-\|x\|)g(x/\|x\|)$ if $\|x\| \geq 1/2$. Then f satisfies the hypotheses of the previous corollary so $f(x) = 0$ for some x in D^n. If $\|x\| \leq 1/2$ then $p = 2x$ is fixed under g and if $\|x\| > 1/2$, $p = x/\|x\|$ in S^{n-1} is mapped by g to λp where $\lambda = (2-2\|x\|)^{-1} > 1$.

As indicated in the opening paragraph, Alexander's Addition Theorem is a generalization of a Phragmen-Brouwer property.

Namely, for Euclidean and other simply connected, locally path connected spaces it implies ($n = 1$) that if neither of two disjoint closed sets separates two points then their union does not. For this reason (and to have a name to distinguish it) the following complementary restatement of the Theorem will be called the Phragmen-Brouwer corollary. It is useful here in obtaining a simple but powerful special case of the Mayer-Vietoris formula for Betti numbers (see [4], page 53). This formula, as noted by Dold [3] appears in many different contexts, e.g. where $k(X)$ denotes cardinality of finite sets, dimension of linear subspaces, Euler characteristic, measure, etc.

Corollary (Phragmen-Brouwer). *If every* $g:B^n \to U_1 \cup U_2$ *is a boundary map* (U_i open), *then* $f:B^{n-1} \to U_1 \cap U_2$ *is a boundary map in* $U_1 \cap U_2$ *if it is a boundary map in each* U_i.

Proof. This follows immediately from the Theorem with $X = U_1 \cup U_2$ and $C_i = X-U_i$ since the extension f_i exists if f is a boundary map in $U_i = X-C_i$ and F exists since $g = f_1 \cup f_2$ is a boundary map in $U_1 \cup U_2 = X-C_1 \cap C_2$ (the conclusion implies f is a boundary map in $X-C_1 \cup C_2 = U_1 \cap U_2$).

Corollary (Mayer-Vietoris). *If every* $g:B^1 \to U_1 \cup U_2$ *is a boundary map and each* U_i *is open and locally path connected, then the components of* $U_1 \cap U_2$ *are the non-void intersections of components of* U_1 *and* U_2. *Also, if* $k(X)$ *is the number of components of* X, *then*

$$\text{(A)} \qquad k(U_1)+k(U_2) = k(U_1 \cup U_2)+k(U_1 \cap U_2).$$

Proof. Ignore components of U_1 missing U_2 (and vice versa) as they contribute nothing to $U_1 \cap U_2$ and equally to both sides of formula (A).

(i) Components W_i of U_i are path connected, so any map $f:B^0 = S^0 \to W_1 \cap W_2 \subset U_1 \cap U_2$ is a boundary map in each U_i. The previous Corollary, with $n = 1$, gives a path P in $U_1 \cap U_2$

between the two points of $f(S^0)$. A connected set in $U_1 \cap U_2$ meeting W_i lies in W_i so $P \subset W_1 \cap W_2$ and $W_1 \cap W_2$ is connected. Similarly, $W_1 \cap W_2$ contains therefore equals the component of $U_1 \cap U_2$ containing $W_1 \cap W_2$.

(ii) If $W_1,\ldots,W_n$ are the components of U_1 and U_2 and p_m is the number of intersecting pairs in $\{W_1,\ldots,W_m\}$, then (i) reduces (A) to $k(\bigcup_{i=1}^{n} W_i) = n-p_n$. Using induction, $k(W_1) = 1 = 1-p_1$ so assume $k(\bigcup_{i=1}^{m} w_i) = m-p_m$ for $1 \leq m < n$. Let V be a component of $\bigcup_{i=1}^{m} W_i$ meeting W_{m+1}, a component of U_1 (similar proof for U_2). The hypotheses of this Corollary still hold if U_2 is replaced by $V \cup U_2$ so by (i) the intersection of W_{m+1} and the component of $V \cup U_2$ containing V is connected. Thus W_{m+1} meets exactly one $W_i \subset V$ $(i \leq m)$ and W_{m+1} "ties together" $p_{m+1}-p_m$ components of $\bigcup_{i=1}^{m} W_i$ to make one of $\bigcup_{i=1}^{m+1} W_i$. Then $k(\bigcup_{i=1}^{m+1} W_i) = m-p_m-(p_{m+1}-p_m)+1 = (m+1) - p_{m+1}$ as required.

A third and final application here of Alexander's Addition is what amounts to a special case of Alexander's Duality Theorem (which is the extension of the Jordan-Brouwer Separation Theorem for which Alexander originally proved his Addition Theorem [1]). It is used in conjunction with the other applications to obtain Jordan Separation theorems. Formula (A) gives a very simple short proof of Jordan Separation for an embedding of S^{m-1} in S^m which is locally flat at one point (in contrast to Dold's requirement that the entire embedding is flat [3]). Using the No Retraction Corollary gives Jordan Separation with no side conditions although no simple way was found for showing there are not more than two complementary components.

Corollary (Alexander Duality). *Every* $f:B^{n-1} \to S^m-D$ *is a boundary map if* D *is a k-disk in* S^m.

Proof (by induction). A 0-disk D is a point and S^m-D is homeomorphic to E^m. In this case, f is a boundary map by the comment preceding the Lemma. Assume the Corollary is true for disks of dimension less than k and let $h:D^k \to D \subset S^m$ be an embedding. Since the image D(s) of the part of D^k in the hyperplane $x_1 = s(|s| \leq 1)$ is a disk of dimension less than k, $f:B^{n-1} \to S^m-D \subset S^m-D(s)$ extends to $f_1:K_1^n \to S^m-D(s)$ where $\partial K_1^n = B^{n-1}$. In fact, since the compact set $h^{-1}f_1(K_1^n)$ misses $x_1 = s$, it misses a "hyperslab" $r \leq x_1 \leq t$ where $r \leq s \leq t$, $r < s$ if $-1 < s$ and $s < t$ if $s < 1$. Then if D(r,t) is the image under h of the part of D^k in this hyperslab (see Figure 4), f is a boundary map in $S^m-D(r,t)$. Choose s to be the least upper bound of all real numbers x in [-1,1] such that f is a boundary map in $S^m-D(-1,x)$. Then $-1 < s$ so $r < s$ and if $X = S^m$, $C_1 = D(r,t)$ and $C_2 = D(-1,r)$, $f_1(K_1^n) \subset X-C_1$ and by choice of s, f also extends to $f_2:K_2^m \to X-C_2$ with $\partial K_2^n = B^{n-1}$ (and we may assume $K_1^n \cap K_2^n = B^{n-1}$). But $C_1 \cap C_2 = D(r)$ is a disk of dimension less than k so $f_1 \cup f_2:K_1^n \cup K_2^n \to X-C_1 \cap C_2 = S^m-D(r)$ extends to $F:K^{n+1} \to X-C_1 \cap C_2$ and Alexander Addition implies f is a boundary map in $X-C_1 \cup C_2 = S^m-D(-1,t)$. This violates the choice of s if $s < 1$, since $s < 1$ implies $s < t$. Thus $s = 1 = t$ and f is a boundary map in $S^m-D(-1,1) = S^m-D$ as required.

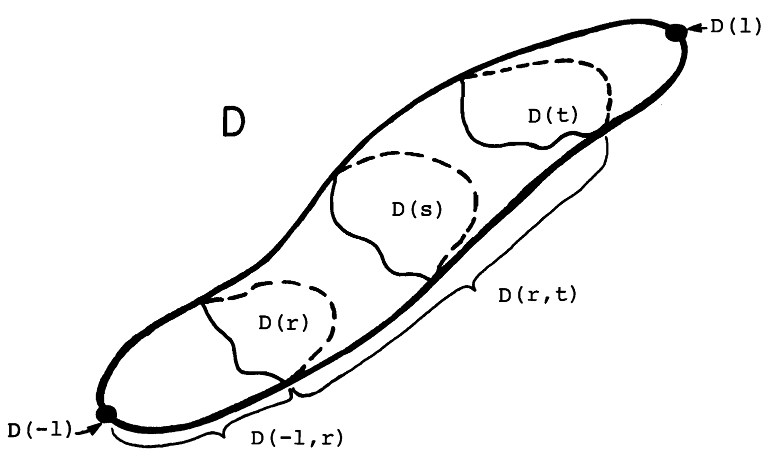

Figure 4

Corollary (Jordan Separation I). *An (m-1)-sphere* S *embedded in* S^m *is the (point set) boundary of each component of* S^m-S.

Proof. If U is a component of S^m-S and N is a neighborhood of a point p of S, there is an (m-1)-disk $D \subset S$ with $S-D \subset N$. The previous result with n = 1, k = m = 1 implies p is joined to any point q of U by a path $A \subset S^m-D$. The component of A-S containing q contains a point of N (else it is a proper open and closed subset of A) hence lies in U. Thus p is in the closure $\overline{U}$ of the (open) component U and $S = \overline{U}-U$ is the boundary of U.

Corollary (Jordan Separation II). *If* S *is an (m-1)-sphere in* S^m *which is locally flat at one point, then* S^m-S *has two components.*

Proof. "Locally flat" means there is a neighborhood N of a point p of S and a homeomorphism $h:N \to E^m$ such that $h(N \cap S)$ is the hyperplane $x_1 = 0$. Let D be an (m-1)-disk as in the previous proof and note that N-D is connected and S-D separates it into two components (images of upper and lower half space of E^m under h^{-1}). Let U_1 = N-D and $U_2 = S^m-S$. By Alexander Duality, every $f:B^{n-1} \to U_1 \cup U_2 = S^m-D$ is a boundary map so (n = 2) formula (A) applies and $k(S^m-S) = k(S^m-D)+k(N-S)-k(N-D) = 1+2-1 = 2$.

Corollary (Jordan Separation III). *An (m-1)-sphere* S *in* S^m *separates* S^m.

Proof. If $f_m:B^o \to S^m-S$ is not a boundary map, then S^m-S has at least two components so the Corollary follows from an inductive proof that for any (n-1)-sphere S in S^m there is a non-boundary map $f_n:B^{m-n} \to S^m-S$, $1 \leq n \leq m$. If n = 1, the two point set S is separated by a hyperplane H. As in the Intermediate Value proof, H contains a cube circumscribing $H \cap S^m$ whose boundary, B^{m-1}, is an (m-1)-sphere. Let L be the line containing S and f_1 be radial projection, from L $\cap$ H, of B^{m-1} onto H $\cap$ $S^m \subset S^m$-S. Then f_1 is a non-boundary map since an extension $F:K^m \to S^m-S$ ($\partial K^m = B^{m-1}$) followed by projection onto H parallel to L and then projection from L $\cap$ H onto B^{m-1} violates the No Retraction Corollary. Suppose $1 \leq n < m$ and S is an n-sphere in S^m. Choose n-disks D_i such that $D_1 \cup D_2 = S$ and $D_1 \cap D_2$ is an (n-1)-sphere in S^m. By induction there is a non-boundary map $f_n:B^{m-n} \to S^m-D_1 \cap D_2$. Alexander Duality implies $D_1 \cap f_n(B^{m-n})$ is non-void. Subdivide B^{m-n} so no simplex meets both sets $f_n^{-1}(D_i)$ and let K_1 (respectively, K_2) be the complex of all (m-n)-simplexes of B^{m-n} missing (meeting) $f_n^{-1}(D_1)$. Then K_1 and K_2 have a common boundary B^{m-n-1} and $f_{n+1} = f_n|B^{m-n-1}$ is a non-boundary map in S^m-S. For if f_{n+1} extends to $F:K \to S^m-S$ with $\partial K = B^{m-n-1}$ (we may assume $\partial K = K \cap B^{m-n}$) then Alexander Duality implies $F \cup (f_n|K_i):K \cup K_i \to S^m-D_i$ extends to $F_i:K_i'$ $\to S^m-D_i$ where $\partial K_i' = K \cup K_i$ (and we may assume $K_1' \cap K_2' = K$). But then $F_1 \cup F_2:K_1' \cup K_2' \to S^m-D_1 \cap D_2$ extends f_n with $\partial(K_1' \cup K_2') = K_1 \cup K_2 = B^{m-n}$ contrary to f_n being a non-boundary map (see Figure 5).

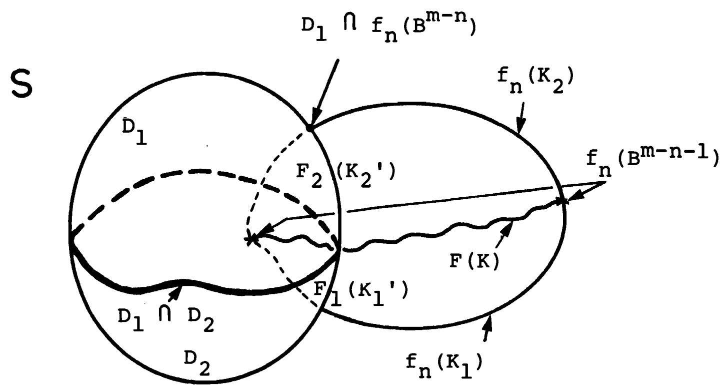

Figure 5

Corollary (Invariance of Domain). *If* U *is open in* S^m *and* h *embeds* U *in* S^m, *then* h(U) *is open.*

Proof. Every point p of U has a closed neighborhood D in U which is an m-disk with (m-1)-sphere boundary S. It suffices to show h(D-S) is open. Let V be the (open) component of S^m-h(S) containing h(p) and W = S^m-h(S)-V. Jordan Separation III implies W is non-void. Since D-S is connected and contains p, h(D-S) $\subset$ V so S^m-h(D) = W $\cup$ (V-h(D)). But S^m-h(D) is connected by Alexander Duality so V-h(D) = V-h(D-S) is empty and h(D-S) = V is open.

Corollary (Invariance of Dimension). *If* $h: S^m \to S^n$ *is a homeomorphism, then* m = n.

Proof. If $n < m$ ($m < n$ is similar) and $i: S^n \to S^m$ is inclusion, then $i \circ h$ is a non-open embedding of the open set U = S^m in S^m contradicting Invariance of Domain.

There are undoubtedly many other simple but significant applications of the results of this paper and there remains the challenge of finding a simple proof that S^m-S (S is an (m-1)-sphere) has at most two components without assuming a flat spot.

References

1. J. W. Alexander, *A proof and extension of the Jordan-Brouwer separation theorem*, Trans. Amer. Math. Soc. 23 (1922), 333-349.
2. R. H. Bing, *The elusive fixed point property*, Amer. Math. Monthly 76 (1969), 119-132.
3. A. Dold, *Einfacher Beweis des Jordanschen Kurvensatzes unter zusätzlichen Voraussetzungen*, Math.-Phys. Semesterber. (Göttingen) 17 (1970), 23-32.
4. S. Eilenberg, and N. Steenrod, *Foundations of algebraic topology*, Princeton University Press, Princeton, 1952.
5. M. H. A. Newman, *Elements of the topology of plane sets of points*, Cambridge University Press, Cambridge, 1939.

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