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# AN ELEMENTARY ROMP THROUGH SOME ADVANCED EUCLIDEAN TOPOLOGY 

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#### Abstract

In a discussion following R. H. Bing's talk on "The elusive fixed point property" [2] at the 1968 Houston Topology Conference it was observed that the easy proof of Brouwer's Fixed Point Theorem for an interval, obtained by considering points which move right and those moving left, extends to two dimensions with the aid of a Phragmen-Brouwer property but no further. The author tried for some time to find the proper extension of this property with little success. Upon reading a new solution of this two-dimensional problem (and others) by Albrecht Dold [3], the proper extension was eventually found to be a variation of what is sometimes called Alexander's Addition Theorem. Since it is really a restatement of the latter, it is not a new result but is a new way of interpreting and relating known results.

This new approach permits very short simple proofs, using only a few simple concepts (finite Euclidean complexes, mappings, compactness and connectedness properties), of a large number of powerful theorems in Euclidean topology. It most nearly reflects the work of the above-mentioned work of Dold and Chapter Five of M. H. A. Newman's book [5]. Unlike these, it does not explicitly involve any homotopy or homology theory but they are just under the surface (especially the singular theory) so that it could serve as excellent motivation for the introduction of algebraic topology.

The presentation is simplified if we agree to reserve some symbols to represent certain standard objects. Let $\mathrm{E}^{\mathrm{n}}$ be Euclidean $n$-space, $D^{n}$ its unit disk (all $x$ in $E^{n}$ with $\|x\| \leq 1$ )


and $s^{n}$ the unit $n$-sphere (all $x$ in $E^{n+1}$ with $\|x\|=1$ ). Homeomorphs of $\mathrm{D}^{\mathrm{n}}$ and $\mathrm{S}^{\mathrm{n}}$ will be called $n$-disks and $n$-spheres, respectively. An $n$-simplex, $\sigma^{n}$, is the convex hull of $n+1$ independent points of $E^{n}$, called vertices of $\sigma^{n}$, and the convex hull of any subset of vertices is called a face of $\sigma^{n}$. Let $K^{n}$ denote an $n$-complex, a finite union of $n$-simplexes which are pairwise disjoint or meet in a common face ( $\mathrm{K}^{\mathrm{n}}$ is properly a pair consisting of a collection of $n$-simplexes and their union). The ( $\mathrm{n}-1$ )-simplexes which are faces of an odd number of n simplexes of $K^{n}$ form an ( $\mathrm{n}-1$ )-complex $\partial \mathrm{K}^{\mathrm{n}}$ called the (mod 2) boundary of $\mathrm{K}^{\mathrm{n}}$. An n -complex with empty boundary is called an $n$-boundary and denoted $B^{n}$. A map (continuous function) $f: B^{n} \rightarrow X$ is a boundary map (in $A$ ) if it extends to a map $F: K^{n+1} \rightarrow X$ where $\partial \mathrm{K}^{\mathrm{n}+1}=\mathrm{B}^{\mathrm{n}}\left(\right.$ and $\left.\mathrm{F}\left(\mathrm{K}^{\mathrm{n}+1}\right) \subset \mathrm{A}\right)$. Dimensional superscripts may be omitted if no confusion results.

The only result needed to establish Alexander's Addition Theorem is the familiar fact, proved below, that the boundary of an n-complex is an ( $\mathrm{n}-1$ )-boundary. The converse of this is also true (justifying the terminilogy) since the cone over an ( $\mathrm{n}-1$ )-boundary B is easily seen to be an n -complex with boundary B. Two more comments will make the conditions in the Addition Theorem seem more natural or attainable. First, if two complexes $K_{1}$ and $K_{2}$ have the same boundary $B$, for most purposes it is no loss of generality to assume $K_{1} \cap K_{2}=B$. One need only embed the (finite number of) vertices of $B, K_{1}-B$ and $K_{2}-B$ independently in some $\mathrm{E}^{\mathrm{n}}$ in order to obtain isomorphic (isometric if need be) copies of $B, K_{1}$ and $K_{2}$ with the desired properties. Finally, any map of an n-boundary $B$ into a convex set $A$ is a boundary map in A since it extends to the cone over B (linear on each element of the cone).

Lemma. $\quad \partial \partial K^{\mathrm{n}}=\phi$.

Proof. If $\sigma^{n-2}$ in $\partial K^{n}$ is a face of $m(n-1)-s i m p l e x e s$ $\sigma_{1}, \ldots:, \sigma_{m}$ of $\partial K^{n}$, we need only show $m$ is even. Let $\sigma_{1}^{\prime}, \ldots, \sigma_{m}^{\prime}$, be all other ( $n-1$ )-simplexes of $K^{n}$ with face $\sigma^{n-2}$. By definilion of $\partial K^{n}, \sigma_{i}$ is a face of $2 n_{i}+1$ n-simplexes of $K^{n}$ and $\sigma_{i}^{\prime}$ is a face of $2 n_{i}^{\prime}$ such simplexes. This counts each n-simplex with face $\sigma^{n-2}$ twice (once for each vertex not in $\sigma^{n-2}$, see Figure 1) so $\sum_{i=1}^{m}\left(2 n_{i}+1\right)+\sum_{i=1}^{m^{\prime}} 2 n_{i}^{\prime}=2\left(\sum_{i=1}^{m} n_{i}+\sum_{i=1}^{m^{\prime}} n_{i}^{\prime}\right)+m$ is even which implies $m$ is even.


Figure 1

Theorem (Alexander Addition). For $i=1,2$ suppose $C_{i}$ is closed in $X, B^{n-1}=\partial K_{i}^{n}=K_{1}^{n} \cap K_{2}^{n}$ and $K_{1}^{n} U K_{2}^{n}=\partial K^{n+1}$. If $\mathrm{f}: \mathrm{B}^{\mathrm{n}-1} \rightarrow \mathrm{X}-\mathrm{C}_{1} \cup \mathrm{C}_{2}$ extends to $\mathrm{f}_{\mathrm{i}}: \mathrm{K}_{\mathrm{i}}^{\mathrm{n}} \rightarrow \mathrm{X}-\mathrm{C}_{\mathrm{i}}$ and $\mathrm{f}_{1} \cup \mathrm{f}_{2}$ extends to $\mathrm{F}: \mathrm{K}^{\mathrm{n}+1} \rightarrow \mathrm{X}-\mathrm{C}_{1} \cap \mathrm{C}_{2}$, then for some $\mathrm{K}^{\mathrm{n}} \subset \mathrm{K}^{\mathrm{n}+1}, \partial \mathrm{~K}^{\mathrm{n}}=\mathrm{B}^{\mathrm{n}-1}$ and $\mathrm{F}\left(\mathrm{K}^{\mathrm{n}}\right) \subset \mathrm{X}-\mathrm{C}_{1} \cup \mathrm{C}_{2}$.

Proof. Since $F^{-1}\left(C_{i}\right)$ is compact and $F^{-1}\left(C_{1}\right) \cap\left(F^{-1}\left(C_{2}\right) \cup K_{1}^{n}\right)=$ $\left(F^{-1}\left(C_{1}\right) \cap F^{-1}\left(C_{2}\right)\right) U\left(F^{-1}\left(C_{1}\right) \cap K_{1}^{n}\right)=F^{-1}\left(C_{1} \cap C_{2}\right) U f_{1}^{-1}\left(C_{1}\right)=\phi$, then $K^{n+1}$ can be subdivided so that the complex $K$ of ( $n+1$ )simplexes meeting $F^{-1}\left(C_{2}\right) U K_{1}^{n}$ misses $F^{-1}\left(C_{1}\right)$. Then $\partial K=K_{l}^{n} U K^{n}$ where $K^{n}$ misses $F^{-1}\left(C_{1}\right) U F^{-1}\left(C_{2}\right)$, i.e. $F\left(K^{n}\right) \subset X-C_{1} U C_{2}\left(K_{1}^{n}\right.$ and $K^{n}$ have no common $n$-simplex, see Figure 2). The Lemma gives $\partial\left(K_{1}^{n} U K^{n}\right)=\partial \partial K=\phi$ so if $\sigma^{n-1}$ is a face of $m$ n-simplexes of $K^{n}$ and $m^{\prime} n$-simplexes of $K_{1}^{n}$, then $m+m^{\prime}$ is even. Thus $m$ is odd
iff $m^{\prime}$ is odd and $\partial K^{n}=\partial K_{1}^{n}=B^{n-1}$.


Figure 2
Corozzary (No Retraction). No complex retracts to an
n-sphere which is its (mod 2) boundary.
Proof (by induction). If $\partial K^{l}$ is a 0 -sphere, $K^{1}$, contains an arc joining the two points of $\partial K^{l}$ which, being connected, cannot be retracted to the disconnected set $\partial K^{l}$. Suppose $n>0$ and the corollary is true for values less than $n$. If $F$ retracts $X=K^{n+1}$ to $S=\partial K^{n+1}$, an $n-s p h e r e, ~ l e t ~ C_{1}$ be all of $s$ except for the interior of an $n$-simplex and $C_{2}$ a single point in $S-C_{1}$. Subdivide $K^{n+1}$ so there is an $n-s i m p l e x$ in $S-C_{1}$ with $C_{2}$ in its interior, let $K_{1}^{n}$ be this inner $n-s i m p l e x$ and $K_{2}^{n}$ the union of all other $n-s i m p l e x e s$ of $\partial K^{n+1}$ (see Figure 3). If $f$ and $f i$ are the identity on $\partial K_{1}^{n}=\partial K_{2}^{n}$ and $K_{i}^{n}$ respectively, then Alexander's Addition gives a complex $K^{n}$ with $\partial K^{n}=\partial K_{1}^{n}$ (an ( $n-l$ )-sphere) and $F\left(K^{n}\right) \subset X-C_{1} \cup C_{2}$. But then $F$ followed by projection from $C_{2}$ retracts $K^{n}$ to $\partial K^{n}$ contrary to the induction assumption.


Figure 3

Corozzary (Intermediate value). If a map $f: D^{n} \rightarrow E^{n}$ is the identity on $\mathrm{S}^{\mathrm{n}-1}$ then $\mathrm{D}^{\mathrm{n}} \subset \mathrm{f}\left(\mathrm{D}^{\mathrm{n}}\right)$, i.e. f takes on every "value" in $D^{n}$.

Proof. If $p$ is a point of $D^{n}$ not in $f\left(D^{n}\right)$, let $K^{n}$ be the cube bounded by the planes $x_{i}= \pm 1$ which circumscribes $D^{n}$ and $r$ the radial homeomorphism shrinking $K^{n}$ to $D^{n}$. Then for followed by projection on the $(n-1)-s p h e r e ~ \partial K^{n}$ from $p$ is a retraction contradicting the previous Corollary.

Corolzary (Brouwer Fixed Point). Every map g:D ${ }^{\mathbf{n}} \rightarrow \mathrm{E}^{\mathbf{n}}$ either leaves a point of $D^{n}$ fixed or stretches a point of $s^{n-1}$. That is, for some $p \in D^{n}, g(p)=\lambda p, \lambda \geq 1$ and $p \in s^{n-1}$ if $\lambda>1$.

Proof. Let $f(x)=2 x-g(2 x)$ if $\|x\| \leq 1 / 2$ and $f(x)=$ $x /\|x\|-2(1-\|x\|) g(x /\|x\|)$ if $\|x\| \geq 1 / 2$. Then $f$ satisfies the hypotheses of the previous corollary so $f(x)=0$ for some $x$ in $D^{n}$. If $\|x\| \leq 1 / 2$ then $p=2 x$ is fixed under $g$ and if $\|x\|>1 / 2, p=x /\|x\|$ in $s^{n-1}$ is mapped by $g$ to $\lambda p$ where $\lambda=(2-2\|x\|)^{-1}>1$.

As indicated in the opening paragraph, Alexander's Addition Theorem is a generalization of a Phragmen-Brouwer property.

Namely, for Euclidean and other simply connected, locally path connected spaces it implies ( $\mathrm{n}=1$ ) that if neither of two disjoint closed sets separates two points then their union does not. For this reason (and to have a name to distinguish it) the following complementary restatement of the Theorem will be called the Phragmen-Brouwer corollary. It is useful here in obtaining a simple but powerful special case of the Mayer-Vietoris formula for Betti numbers (see [4], page 53). This formula, as noted by Dold [3] appears in many different contexts, e.g. where $k(X)$ denotes cardinality of finite sets, dimension of linear subspaces, Euler characteristic, measure, etc.

Corollary (Phragmen-Brouwer). If every $\mathrm{g}: \mathrm{B}^{\mathrm{n}} \rightarrow \mathrm{U}_{1} \mathrm{U} \mathrm{U}_{2}$ is a boundary map ( $\mathrm{U}_{\mathrm{i}}$ open), then $\mathrm{f}: \mathrm{B}^{\mathrm{n-1}} \rightarrow \mathrm{U}_{1} \cap \mathrm{U}_{2}$ is a boundary map in $\mathrm{U}_{1} \cap \mathrm{U}_{2}$ if it is a boundary map in each $\mathrm{U}_{\mathrm{i}}$.

Proof. This follows immediately from the Theorem with $X=U_{1} U U_{2}$ and $C_{i}=X-U_{i}$ since the extension $f_{i}$ exists if $f$ is a boundary map in $U_{i}=X-C_{i}$ and $F$ exists since $g=f_{1} U f_{2}$ is a boundary map in $U_{1} U U_{2}=x-C_{1} \cap C_{2}$ (the conclusion implies $f$ is a boundary map in $X-C_{1} \cup C_{2}=U_{1} \cap U_{2}$ ).

Corollary (Mayer-Vietoris). If every $\mathrm{g}: \mathrm{B}^{\mathrm{l}} \rightarrow \mathrm{U}_{1} \mathrm{U}_{\mathrm{U}}$ is a boundary map and each $\mathrm{U}_{\mathrm{i}}$ is open and locally path connected, then the components of $\mathrm{U}_{1} \cap \mathrm{U}_{2}$ are the non-void intersections of components of $\mathrm{U}_{1}$ and $\mathrm{U}_{2}$. Also, if $\mathrm{k}(\mathrm{X})$ is the number of components of x , then

$$
\begin{equation*}
k\left(U_{1}\right)+k\left(U_{2}\right)=k\left(U_{1} U U_{2}\right)+k\left(U_{1} \cap U_{2}\right) \tag{A}
\end{equation*}
$$

Proof. Ignore components of $\mathrm{U}_{1}$ missing $\mathrm{U}_{2}$ (and vice versa) as they contribute nothing to $U_{1} \cap U_{2}$ and equally to both sides of formula (A).
(i) Components $W_{i}$ of $U_{i}$ are path connected, so any map $f: B^{\circ}=s^{\circ} \rightarrow W_{1} \cap W_{2} \subset U_{1} \cap U_{2}$ is a boundary map in each $U_{i}$. The previous Corollary, with $n=1$, gives a path $P$ in $U_{1} \cap U_{2}$
between the two points of $f\left(S^{\circ}\right)$. A connected set in $U_{1} \cap U_{2}$ meeting $W_{i}$ lies in $W_{i}$ so $P \subset W_{1} \cap W_{2}$ and $W_{1} \cap W_{2}$ is connected. Similarly, $W_{1} \cap W_{2}$ contains therefore equals the component of $\mathrm{U}_{1} \cap \mathrm{U}_{2}$ containing $\mathrm{W}_{1} \cap \mathrm{~W}_{2}$.
(ii) If $W_{1}, \ldots, W_{n}$ are the components of $U_{1}$ and $U_{2}$ and $p_{m}$ is the number of intersecting pairs in $\left\{W_{1}, \ldots, W_{m}\right\}$, then (i) reduces ( $A$ ) to $k\left(U \underset{i=1}{n} W_{i}\right)=n-p_{n}$. Using induction, $k\left(W_{1}\right)=1=$ $1-p_{1}$ so assume $k\left(U{ }_{i=1}^{m} w_{i}\right)=m-p_{m}$ for $l \leq m<n$. Let $v$ be a component of $U{ }_{i=1}^{m} W_{i}$ meeting $W_{m+1}$, a component of $U_{1}$ (similar proof for $U_{2}$ ). The hypotheses of this Corollary still hold if $U_{2}$ is replaced by $v U U_{2}$ so by (i) the intersection of $W_{m+1}$ and the component of $\mathrm{V} \mathrm{U} \mathrm{U} \mathrm{U}_{2}$ containing V is connected. Thus $\mathrm{W}_{\mathrm{m}+1}$ meets exactly one $\mathrm{W}_{\mathrm{i}} \subset \mathrm{V}(\mathrm{i} \leq \mathrm{m})$ and $\mathrm{W}_{\mathrm{m}+\mathrm{l}}$ "ties together" $p_{m+1}-p_{m}$ components of $U \underset{i=1}{m} W_{i}$ to make one of $U \underset{i=1}{m+1} W_{i}$. Then $k\left(U \underset{i=1}{m+1} W_{i}\right)=m-p_{m}-\left(p_{m+1}-p_{m}\right)+1=(m+1)-p_{m+1}$ as required.

A third and final application here of Alexander's Addition is what amounts to a special case of Alexander's Duality Theorem (which is the extension of the Jordan-Brouwer Separation Theorem for which Alexander originally proved his Addition Theorem [l]). It is used in conjunction with the other applications to obtain Jordan Separation theorems. Formula(A)gives a very simple short proof of Jordan Separation for an embedding of $s^{m-1}$ in $s^{m}$ which is locally flat at one point (in contrast to Dold's requirement that the entire embedding is flat [3]). Using the No Retraction Corollary gives Jordan Separation with no side conditions although no simple way was found for showing there are not more than two complementary components.

Corollary (Alexander Duality). Every $\mathrm{f}: \mathrm{B}^{\mathrm{n}-1} \rightarrow \mathrm{~S}^{\mathrm{m}}-\mathrm{D}$ is a boundary map if D is a $k$-disk in $\mathrm{s}^{\mathrm{m}}$.

Proof (by induction). A 0 -disk $D$ is a point and $s^{m}-D$ is homeomorphic to $\mathrm{E}^{\mathrm{m}}$. In this case, f is a boundary map by the
comment preceding the Lemma. Assume the Corollary is true for disks of dimension less than $k$ and let $h: D^{k} \rightarrow D \subset s^{m}$ be an embedding. Since the image $D(s)$ of the part of $D^{k}$ in the hyperplane $x_{1}=s(|s| \leq 1)$ is a disk of dimension less than $k$, $f: B^{n-1} \rightarrow s^{m}-D \subset s^{m}-D(s)$ extends to $f_{1}: K_{1}^{n} \rightarrow s^{m}-D(s)$ where $\partial K_{1}^{n}=B^{n-1}$. In fact, since the compact set $h^{-l} f_{1}\left(K_{1}^{n}\right)$ misses $\mathrm{x}_{1}=\mathrm{s}$, it misses a "hyperslab" $\mathrm{r} \leq \mathrm{x}_{1} \leq t$ where $\mathrm{r} \leq \mathrm{s} \leq \mathrm{t}$, $\mathrm{r}<\mathrm{s}$ if $-1<s$ and $s<t$ if $s<l$. Then if $D(r, t)$ is the image under $h$ of the part of $D^{k}$ in this hyperslab (see Figure 4), $f$ is a boundary map in $s^{m}-D(r, t)$. Choose $s$ to be the least upper bound of all real numbers $x$ in $[-1,1]$ such that $f$ is a boundary map in $s^{m}-D(-1, x)$. Then $-1<s$ so $r<s$ and if $x=s^{m}, C_{1}=D(r, t)$ and $C_{2}=D(-1, r), f_{1}\left(K_{1}^{n}\right) \subset x-C_{1}$ and by choice of $s, f$ also extends to $f_{2}: K_{2}^{m} \rightarrow X-C_{2}$ with $\partial K_{2}^{n}=B^{n-l}$ (and we may assume $\mathrm{K}_{1}^{\mathrm{n}} \cap \mathrm{K}_{2}^{\dot{\mathrm{n}}}=\mathrm{B}^{\mathrm{n}-1}$ ). But $\mathrm{C}_{1} \cap \mathrm{C}_{2}=\mathrm{D}(\mathrm{r})$ is a disk of dimension less than $k$ so $f_{1} \cup f_{2}: K_{1}^{n} U K_{2}^{n} \rightarrow X-C_{1} \cap C_{2}=s^{m}-D(r)$ extends to $\mathrm{F}: \mathrm{K}^{\mathrm{n}+1} \rightarrow \mathrm{X}-\mathrm{C}_{1} \cap \mathrm{C}_{2}$ and Alexander Addition implies f is a boundary map in $x-C_{1} \cup C_{2}=s^{m}-D(-1, t)$. This violates the choice of $s$ if $s<1$, since $s<l$ implies $s<t$. Thus $s=1=t$ and $f$ is a boundary map in $s^{m}-D(-1,1)=s^{m}-D$ as required.


Corollary (Jordan Separation I). An (m-1)-sphere S embedded in $S^{m}$ is the (point set) boundary of each component of $s^{m}-S$.

Proof. If $U$ is a component of $S^{m}-S$ and $N$ is a neighborhood of a point $p$ of $S$, there is an (m-l)-disk $D \subset S$ with $S-D \subset N . \quad$ The previous result with $\mathrm{n}=\mathrm{l}, \mathrm{k}=\mathrm{m}=\mathrm{l}$ implies. $p$ is joined to any point $q$ of $U$ by a path $A \subset s^{m}-D$. The component of A-S containing $q$ contains a point of $N$ (else it is a proper open and closed subset of A) hence lies in U. Thus $p$ is in the closure $\bar{U}$ of the (open) component $U$ and $S=\bar{U}-U$ is the boundary of $U$.

Corollary (Jordan Separation II). If S is an (m-1)-sphere in $s^{\mathrm{m}}$ which is locally flat at one point, then $s^{m}-\mathrm{s}$ has two components.

Proof. "Locally flat" means there is a neighborhood N of a point $p$ of $S$ and a homeomorphism $h: N \rightarrow E^{m}$ such that $h(N \cap S)$ is the hyperplane $x_{1}=0$. Let $D$ be an ( $m-1$ ) - disk as in the previous proof and note that $N-D$ is connected and $S-D$ separates it into two components (images of upper and lower half space of $E^{m}$ under $h^{-l}$ ). Let $U_{1}=N-D$ and $U_{2}=S^{m}-S$. By Alexander Duality, every $f: B^{n-1} \rightarrow U_{1} U U_{2}=S^{m}-D$ is a boundary map so $\left(\mathrm{n}=2\right.$ ) formula(A) applies and $k\left(S^{m}-S\right)=k\left(S^{m}-D\right)+k(N-S)-k(N-D)=$ $1+2-1=2$.

Corozlary (Jordan Separation III). An (m-1)-sphere $S$ in $S^{m}$ separates $S^{m}$.

Proof. If $f_{m}: B^{\circ} \rightarrow S^{m}-S$ is not a boundary map, then $S^{m}-S$ has at least two components so the Corollary follows from an inductive proof that for any $(n-l)-s p h e r e s$ in $s^{m}$ there is a non-boundary map $f_{n}: B^{m-n} \rightarrow S^{m}-S, l \leq n \leq m$. If $n=1$, the two point set $S$ is separated by a hyperplane $H$. As in the Intermediate Value proof, $H$ contains a cube circumscribing $H \cap s^{m}$ whose boundary, $B^{m-1}$, is an ( $m-1$ )-sphere. Let $L$ be the line
containing $S$ and $f_{1}$ be radial projection, from $L \cap H$, of $B^{m-1}$ onto $H \cap s^{m} \subset s^{m}-S$. Then $f_{1}$ is a non-boundary map since an extension $F: K^{m} \rightarrow S^{m}-S\left(\partial K^{m}=B^{m-1}\right)$ followed by projection onto $H$ parallel to $L$ and then projection from $L \cap H$ onto $B^{m-1}$ violates the No Retraction Corollary. Suppose $1 \leq n<m$ and $S$ is an $n$-sphere in $S^{m}$. Choose $n$-disks $D_{i}$ such that $D_{1} \cup D_{2}=S$ and $D_{1} \cap D_{2}$ is an ( $n-1$ )-sphere in $s^{m}$. By induction there is a non-boundary map $f_{n}: B^{m-n} \rightarrow S^{m}-D_{1} \cap D_{2}$. Alexander Duality implies $D_{1} \cap f_{n}\left(B^{m-n}\right)$ is non-void. Subdivide $B^{m-n}$ so no simplex meets both sets $f_{n}^{-1}\left(D_{i}\right)$ and let $K_{1}$ (respectively, $K_{2}$ ) be the complex of all (m-n)-simplexes of $B^{m-n}$ missing (meeting) $f_{n}^{-1}\left(D_{1}\right)$. Then $K_{1}$ and $K_{2}$ have a common boundary $B^{m-n-1}$ and $f_{n+1}=f_{n} \mid B^{m-n-1}$ is a non-boundary map in $S^{m}$-S. For if $f_{n+1}$ extends to $F: K \rightarrow S^{m}-S$ with $\partial K=B^{m-n-1}$ (we may assume $\partial K=K \cap B^{m-n}$ ) then Alexander Duality implies $F U\left(f_{n} \mid K_{i}\right): K U K_{i} \rightarrow S^{m}-D_{i}$ extends to $F_{i}: K_{i}^{\prime}$ $\rightarrow s^{m}-D_{i}$ where $\partial K_{i}^{\prime}=K \cup K_{i}$ (and we may assume $K_{1}^{\prime} \cap K_{2}^{\prime}=k$ ). But then $F_{1} \cup F_{2}: K_{1}^{\prime} U K_{2}^{\prime} \rightarrow S^{m}-D_{1} \cap D_{2}$ extends $f_{n}$ with $\partial\left(K_{1}^{\prime} U K_{2}^{\prime}\right)=$ $K_{1} \cup K_{2}=B^{m-n}$ contrary to $f_{n}$ being a non-boundary map (see Figure 5).


Corollary (Invariance of Domain). If U is open in $\mathrm{S}^{\mathrm{m}}$ and h embeds U in $\mathrm{S}^{\mathrm{m}}$, then $\mathrm{h}(\mathrm{U})$ is open.

Proof. Every point p of U has a closed neighborhood D in U which is an m-disk with (m-1)-sphere boundary S. It suffices to show $h(D-S)$ is open. Let $V$ be the (open) component of $s^{m}-h(S)$ containing $h(p)$ and $W=S^{m}-h(S)-V$. Jordan Separation III implies $W$ is non-void. Since $D-S$ is connected and contains $p, h(D-S) \subset V$ so $S^{m}-h(D)=W U(V-h(D))$. But $S^{m}-h(D)$ is connected by Alexander Duality so $V-h(D)=V-h(D-S)$ is empty and $h(D-S)=V$ is open.

Corolzary (Invariance of Dimension). If $\mathrm{h}: \mathrm{S}^{\mathrm{m}}+\mathrm{S}^{\mathrm{n}}$ is a homeomorphism, then $\mathrm{m}=\mathrm{n}$.

Proof. If $n<m\left(m<n\right.$ is similar) and i: $S^{n} \rightarrow S^{m}$ is inclusion, then ioh is a non-open embedding of the open set $U=s^{m}$ in $s^{m}$ contradicting Invariance of Domain.

There are undoubtedly many other simple but significant applications of the results of this paper and there remains the challenge of finding a simple proof that $s^{m}-s$ ( $s$ is an (m-l)-sphere) has at most two components without assuming a flat spot.

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