CONTINUA AND CONTINUOUS TRANSFORMATIONS

by

A. Lelek

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This lecture[1] is designed to be a survey of some results,
new and not so new, in Continua Theory with an emphasis on
classification problems and special properties of continuous
mappings. The selection of topics could, as usual, be highly
subjective, reflecting the author's inclinations and tastes.
Nevertheless, an effort has been made to select those themes
which either seem to be significant or offer possibilities of
further development.

1. Transformations

The continuity of a transformation is a condition which
deals with closed subsets of the range space and the closedness
of their preimages in the domain space. Studying continua,
however, requires a combination of two properties: closedness
and connectedness. A direct analogue of continuity here makes
an obviously too restrictive condition, and the way to obtain
interesting useful classes of transformations is only by some-
what relaxing the connectedness of the preimages of connected
closed subsets. Recall that a topological space X is said to
be *connected between* two subsets $A, B \subset X$ provided $X \neq U \cup V$,
where $U, V \subset X$ are open, $U \cap V = \emptyset$ and $A \subset U$, $B \subset V$. Let $f: X \to Y$
be a continuous mapping of a topological space X onto a topo-
logical space Y. The mapping f is called *confluent* [5], *weakly
confluent*, or *pseudo-confluent* [8] provided, for each connected
closed non-empty set $C \subset Y$, the following conditions are satis-
fied, respectively:

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[1]The Third Annual Ralph B. Bennett Memorial Lecture delivered
at the Auburn University Topology Conference on March 18,
1976.

(c) for each pair of points $x \in f^{-1}(C)$ and $y \in C$, the set $f^{-1}(C)$ is connected between $\{x\}$ and $f^{-1}(y)$;

(w) there exists a point $x_o \in f^{-1}(C)$ such that, for each point $y \in C$, the set $f^{-1}(C)$ is connected between $\{x_o\}$ and $f^{-1}(y)$;

(p) for each pair of points $y,y' \in C$, the set $f^{-1}(C)$ is connected between $f^{-1}(y)$ and $f^{-1}(y')$.

Clearly, (c) implies (w), and (w) implies (p). If $f^{-1}(y)$ is compact for each $y \in Y$, then conditions (c), (w), (p) are equivalent to the following conditions, respectively:

(c') for each quasi-component Q of $f^{-1}(C)$, we have $C = f(Q)$;

(w') there exists a quasi-component Q of $f^{-1}(C)$ such that $C = f(Q)$;

(p') for each pair of points $y,y' \in C$, there exists a quasi-component Q of $f^{-1}(C)$ such that $y,y' \in f(Q)$.

Moreover, if X is compact, then "quasi-component" in (c'), (w'), (p') can be replaced by "component." By a *continuum* we understand a connected compact metric space.

Theorem 1 (Nadler [14]). *If X and Y are continua, then the space of confluent mappings of X onto Y with the uniform convergence topology is topologically complete.*

In [14], it is proved that the space of confluent mappings of X onto Y is a G_δ-subset of the space of weakly confluent mappings of X onto Y, and that the latter space is a closed subspace of the space of all continuous mappings of X onto Y with the uniform convergence topology. Also, the space of confluent mappings is not necessarily closed in the space of all continuous mappings. It is closed, however, if some conditions are imposed on Y, e.g., if Y is locally connected [9] or, more generally, if Y has Kelley's property κ [14]. The space of pseudo-confluent mappings of X onto Y is always a closed subspace of the space Y^X of all continuous mappings [9].

2. Classifications

The best known is a classification of continua by means
of ε-mappings. Let ε > 0 be a number. An ε-*mapping* is a con-
tinuous mapping whose point-inverses all have diameters less
than ε. Given a class Σ of topological spaces, a class of con-
tinua, called Σ-*like* continua, is introduced by taking all con-
tinua X such that, for each number ε > 0, there exists an ε-
mapping of X onto an element of Σ. If Σ is the class of all
trees (that is, spaces homeomorphic to 1-dimensional acyclic
connected polyhedra) or the class of all arcs, we get *tree-like*
continua or *arc-like* continua, respectively. The easiest ex-
ample of a tree which is not an arc is, of course, a *simple
triod*, i.e., the union of three arcs having a common end-point
and pairwise disjoint outside that point. Then arcs are exactly
those trees which do not contain any simple triods. It turns
out, however, that the situation is quite different if arcs
and trees are generalized to arc-like continua and tree-like
continua, respectively. By a *triod* we mean a continuum which
is the union of three continua having a proper subcontinuum as
their common part and pairwise disjoint outside that subcontinuum.
A continuum is called *atriodic* provided it does not contain any
triod.

Theorem 2 (Ingram [4]). *There exists an uncountable col-
lection of pairwise disjoint atriodic tree-like continua on
the plane such that none of them is arc-like.*[2]

Theorem 3 (McLean [12]). *Each confluent image of a tree-
like continuum is tree-like.*

Problem 1. *Is each confluent image of an arc-like*

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[2]Professor W. T. Ingram in his talk presented at this confer-
ence gives some further intriguing properties of the same col-
lection of continua.

continuum arc-like? [6]

In connection with Problem 1, we mention another unsolved problem concerning products of mappings. If $f_i:X_i \to Y_i$ are mappings (i = 1,2), we write $f_1 \times f_2$ to denote the mapping of $X_1 \times X_2$ into $Y_1 \times Y_2$ defined by

$$(f_1 \times f_2)(x_1,x_2) = (f_1(x_1), f_2(x_2)) \quad (x_i \in X_i; \ i = 1,2).$$

Problem 2. Suppose $f_i:X_i \to Y_i$ are confluent mappings and X_i are arc-like continua (i = 1,2). Is the product $f_1 \times f_2$: $X_1 \times X_2 \to Y_1 \times Y_2$ confluent?

The assumption that X_i are arc-like is essential in Problem 2. Maćkowiak [10] has an example of a confluent mapping of a (non-arc-like) continuum whose product with the identity mapping of an arc is not pseudo-confluent. We say that a continuum is *rational* provided it admits a basis of open sets with countable boundaries.

Problem 3. Is each confluent image of a rational continuum rational?[3] [7]

Weakly confluent mappings can destroy the tree-likeness, even of an arc. Without some local conditions like rationality, they can also destroy the atriodicity. For instance, by [3], each dendrite is a weakly confluent image of any continuum which contains a copy of the product of an arc and the Cantor set. A continuum is called *acyclic* provided each continuous mapping of it into the circle is homotopic to a constant mapping.

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[3]Professor E. D. Tymchatyn in a letter to the author, dated March 23, 1976, outlines a construction of a rational continuum on the plane which seems to be a good candidate for a counter-example to Problem 3. His example is very much non-atriodic and non-acyclic. Perhaps then, confluent mappings would preserve the rationality of continua if the continua were assumed to be acyclic or, say, arc-like as in Problem 2.

Theorem **4** (Cook; see [8]). *Each pseudo-confluent image of a 1-dimensional acyclic continuum is at most 1-dimensional.*

Theorem 5 (Lelek [5]). *Each confluent image of an acyclic continuum is acyclic.*

There are two more methods of classification of continua. The first method introduces classes of continua by means of the existence of continuous mappings of certain objects. The second method does it by means of the existence of some mappings onto certain objects. The class of locally connected continua is characterized by the first method, namely, these continua coincide with continuous images of the arc. Continuous images of the pseudo-arc have also been investigated. We quote a few theorems which have some bearing on the second method. We say that a continuum is *Suslinian* provided each collection of pairwise disjoint non-degenerate subcontinua of it is countable. Obviously, each rational continuum is Suslinian, but there exist Suslinian continua that are not rational.

Theorem 6 (Cook and Lelek [3]). *A continuum* X *is atriodic and Suslinian if and only if no mapping of* X *onto a simple triod is weakly confluent.*

Theorem 7 (Rogers [15]). *Each indecomposable continuum can be mapped continuously onto Knaster's indecomposable continuum* K.

Theorem 8 (Bellamy [1]). *Each hereditarily indecomposable continuum can be mapped continuously onto the pseudo-arc.*

The continuum K in Theorem 7 is a well-known standard example of an indecomposable arc-like continuum. One composant of K is a one-to-one continuous image of a half-open segment of

the real line, and the remaining composants are one-to-one con-
tinuous images of an open segment. A class of indecomposable
arc-like continua is described in [15] with the property that
each of its elements is a continuous image of each indecomposable
continuum, and K belongs to this class. Again, a class of arc-
like continua is described in [1] with the property that each of
its elements is a continuous image of each hereditarily inde-
composable continuum, and the pseudo-arc belongs to it.

3. Three results

The following results are particularly interesting because
of the techniques used in their proofs. By a λ-*dendroid* we
understand a hereditarily decomposable tree-like continuum.

Theorem 9 (Mańka [11]). *If* X *is a* λ-*dendroid and* F:X → X
is an upper semicontinuous continuum-valued function, then F
has a fixed point, i.e., *there exists a point* x_o ∈ X *such that*
x_o ∈ F(x_o).

The proof of Theorem 9 makes use extensively of the theory
of irreducible continua.

Theorem 10 (Mohler [13]). *If* X *is a* 1-*arcwise connected*
continuum and h:X → X *is a homeomorphism, then* h *has a fixed*
point.

In this theorem, measure-theoretic techniques are used in
the proof.

Returning to Knaster's indecomposable continuum K, there is
still an open question of whether the "invisible" composants of
K are homeomorphic to one another. (The "invisible" composants
of K are those which are one-to-one continuous images of an open
segment.) A partial solution is given by Bellamy [2] who provides
a classification of composants of K and shows, among other

things, that, with an exception of two composants, all other
ones are homeomorphic with some composants different from
them, under powers of a standard homeomorphism of K onto itself.
Confluent mappings and some group-theoretic techniques are used
in [2].

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