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by

JAMES KEESLING

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Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings
	Department of Mathematics & Statistics
	Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
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ALGEBRAIC INVARIANTS IN SHAPE THEORY[1]

James Keesling[2]

Introduction

In the study of shape theory one needs a set of invariants which are useful in determining when two spaces are not shape equivalent. The following properties are shape invariants which have been studied: movability and n-movability, the property of being an FAR or an FANR, shape dimension (fundamental dimension), and the property of being imbeddable up to shape in R^n or some other space. Some of the most useful shape invariants have been algebraic ones. Čech homology, Čech cohomology, and the cohomotopy groups are well known shape invariants that have played an important role in shape theory. In addition there are the shape groups $\check{\pi}_n(X,x)$ which are invariants in the pointed shape category.

In the construction of $\check{H}_n(X)$ and $\check{\pi}_n(X,x)$ one takes the inverse system of nerves of the numerable covers of the space X. One then associates with this inverse system the inverse system of groups $\{H_n(N(\mathcal{U})) : \mathcal{U}$ a numerable cover of X$\}$ and $\{\pi_n(N(\mathcal{U}),x_{\mathcal{U}}) : \mathcal{U}$ a numerable cover of X$\}$. The inverse limit of these systems of groups are $\check{H}_n(X)$ and $\check{\pi}_n(X,x)$, respectively. One can define a category, called the category of pro-groups, in which the objects are inverse systems of groups and in which the morphisms are certain equivalence classes of maps of inverse systems. One can use the first part of the Čech construction to assign to each space X (or pointed space (X,x)), an object in the pro-group category $\{H_n(N(\mathcal{U}))\}$ ($\{\pi_n(N(\mathcal{U}),x_{\mathcal{U}})\}$). Continuous maps and shape morphisms induce morphisms in the pro-group

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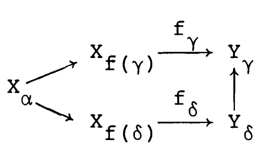
category so that one has functors defined from the category of (pointed) topological spaces and (pointed) continuous maps to the pro-group category so that these functors factor through the shape category. We denote these functors pro-$H_n(X)$ and pro-$\pi_n(X,x)$, respectively.

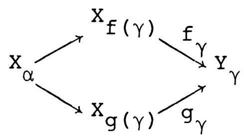
The functors pro-$H_n(X)$ and pro-$\pi_n(X,x)$ have more information about the space X than the associated functors $\check{H}_n(X) = \varprojlim$ pro-$H_n(X)$ and $\check{\pi}_n(X,x) = \varprojlim$ pro-$\pi_n(X,x)$. However, it is more difficult to work with them. It is the purpose of this paper to indicate how the study of the functor pro-$H_n(X)$ for compact spaces X reduces to the study of two functors into the category of abelian groups, $\check{H}_n(X)$ and $\varprojlim^1$ pro-$H_n(X)$. We also give some theorems which show how to compute $\varprojlim^1$ pro-$H_n(X)$ practically. It is also shown that for $n \geq 2$ and a pointed metric continuum (X,x) the study of pro-$\pi_n(X,x)$ reduces to the study of $\check{\pi}_n(X,x)$ and $\varprojlim^1$ pro-$\pi_n(X,x)$. These results have obvious implications in shape theory. It appears likely that they will aid in our understanding of the derived limit functors as well.

1. The Derived Limit Functors

Let C be any category. The category pro-C has for its objects all inverse systems in the category C. If $\{X_\alpha; \pi_{\alpha\beta}; \alpha\leq\beta\in A\}$ and $\{Y_\gamma; \xi_{\gamma\delta}; \gamma\leq\delta\in D\}$ are inverse systems in C (hence objects in pro-C), then we define a *map of systems* $f : \{X_\alpha\} \to \{Y_\gamma\}$ as a pair $(f,\{f_\gamma\})$ where $f : D \to A$ is a function and $\{f_\gamma : \gamma \in D\}$ is a collection of morphisms in C, $f_\gamma : X_{f(\gamma)} \to Y_\gamma$, satisfying the additional condition: If $\gamma \leq \delta$, then there is an $\alpha \in A$ with $\alpha \geq f(\gamma)$, $f(\delta)$ such that the following diagram commutes in C.



If f and g are two maps of systems from $\{X_\alpha\}$ to $\{Y_\gamma\}$, then they are said to be *homotopic*, $f \sim g$, if for each $\gamma \in D$ there is an $\alpha \varepsilon A$, $\alpha \geq f(\gamma), g(\gamma)$ such that the following diagram commutes in C.



The category pro-C has for its morphisms F from $\{X_\alpha\}$ to $\{Y_\gamma\}$ the set of all equivalence classes of maps of systems ($F = [f] : \{X_\alpha\} \to \{Y_\gamma\}$ for some map of systems f).

For the present discussion we are interested in the category of abelian groups and the associated category of pro-abelian groups. The functors pro-$H_n(X)$ and pro-$\pi_n(X,x)$ for $n \geq 2$ mentioned in the Introduction are into this last category.

It is known that if C is an abelian category (for instance the category of abelian groups), then pro-C is also an abelian category (see [2]). Let Ab denote the category of abelian groups and pro-Ab the associated pro-category. Then $\varprojlim$: pro-Ab $\to$ Ab is an additive functor which is left exact. This allows us to define the right derived functors $\varprojlim^0 = \varprojlim$, $\varprojlim^1$, $\varprojlim^2, \ldots$ associated with $\varprojlim$ (see [3] Chapter IV or [7] Chapter XII). Thus we have functors $\varprojlim$ pro-$H_n(x) = \check{H}_n(X)$ and $\varprojlim^k$ pro-$H_n(X)$ for $k \geq 1$. We can also define $\varprojlim^k$ pro-$\pi_n(X,x)$ for $n \geq 2$. For $n = 1$, pro-$\pi_n(X,x)$ may not be an inverse system of abelian groups. One can define derived limits in this case, but they will not be groups in general and one cannot make use of the theory of abelian categories. Abelian categories proved very valuable in our investigation.

There are two questions which must naturally occur to the reader at this point. Associated with any inverse system of abelian groups $\{H_\alpha\}$ are the derived limit groups $\varprojlim^k \{H_\alpha\}$.

Question 1: If we consider $\{H_\alpha\}$ *as an object in* pro-Ab, *how much information about* $\{H_\alpha\}$ *is contained in the sequence of groups* $\{\varprojlim^k \{H_\alpha\}\}$*?*

The definition of the derived limits does not lend itself to easy computation of the groups $\varprojlim^k \{H_\alpha\}$ directly. Theorems which would make it possible to compute and work with these groups would certainly be helpful.

Question 2: How can one effectively compute $\varprojlim^k \{H_\alpha\}$*?*

In this paper we summarize the result of our investigation of these two questions. We obtain satisfactory answers to both questions as far as applications in shape theory for compact spaces are concerned. The results will likely have applications in other areas as well since the derived limit functors are widely used in algebraic topology. They have also been used recently in certain areas of geometric topology.

2. The Isomorphism Theorems

In this section our basic theorems are 2.4 and 2.7. Theorem 2.4 tells us that in shape theory essentially all information about the functor pro-$H_n(X)$ into pro-Ab is contained in the functors $\check{H}_n(X)$ and $\varprojlim^1$ pro-$H_n(X)$ for X compact. Theorem 2.7 tells us that for (X,x) a pointed metric continuum and $n \geq 2$, pro-$\pi_n(X,x)$ is essentially equivalent to the two functors $\check{\pi}_n(X,x)$ and $\varprojlim^1$ pro-$\pi_n(X,x)$. As far as shape theory is concerned these are probably the best answers one could hope for as the answer to Question 1. The answer to Question 2 is given in Section 3.

There has been interest in when $\varprojlim^k \{H_\alpha\} = 0$ for an inverse system of abelian groups. B. Osofski's survey article summarizes some results of this type. The main result is that if the indexing set A has card $A \leq \aleph_n$, then $\varprojlim^k \{H_\alpha\} = 0$ for

$k \geq n + 2$. In particular we have the following theorem.

2.1. *Theorem: If* $\{H_\alpha; \pi_{\alpha\beta}; \alpha \leq \beta \in A\}$ *is an inverse system of abelian groups such that there is a countable cofinal subset of* A, *then* $\varprojlim^k \{H_\alpha\} = 0$ *for* $k \geq 2$.

If we restrict our inverse system to abelian groups of a certain type, then we can sometimes obtain results about the vanishing of $\varprojlim^k$. In particular we have the following two results which can be found in C. Jensen [4].

2.2. *Theorem: If* $\{H_\alpha; \pi_{\alpha\beta}; \alpha \leq \beta \in A\}$ *consists of compact Hausdorff abelian topological groups and continuous homomorphisms, then* $\varprojlim^k \{H_\alpha\} = 0$ *for* $k \geq 1$.

Thus if H_α is finite for all $\alpha \in A$, then $\varprojlim^k \{H_\alpha\} = 0$ for all $k \geq 1$.

2.3. *Theorem: If* H_α *is a finitely generated abelian group for all* $\alpha \in A$, *then* $\varprojlim^k \{H_\alpha\} = 0$ *for all* $k \geq 2$.

Thus if $\{H_\alpha\}$ is a countable inverse system or consists of finitely-generated abelian groups, then no information about the system $\{H_\alpha\}$ is contained in $\varprojlim^k \{H_\alpha\}$ for $k \geq 2$.

2.4. *Theorem: Let* $\{H_\alpha\}$ *and* $\{G_\beta\}$ *be inverse systems of finitely-generated abelian groups. Let* $F : \{H_\alpha\} \to \{G_\beta\}$ *be a pro-homomorphism. Then if* (1) $F_* : \varprojlim \{H_\alpha\} \to \varprojlim \{G_\beta\}$ *is an isomorphism and* (2) $F^1_* : \varprojlim^1 \{H_\alpha\} \to \varprojlim^1 \{G_\beta\}$ *is an isomorphism, then* F *is an isomorphism in* pro-Ab.

2.5. *Corollary: If* $\{H_\alpha\}$ *is an inverse system of finitely-generated abelian groups with* $\varprojlim \{H_\alpha\} = 0$ *and* $\varprojlim^1 \{H_\alpha\} = 0$, *then* $\{H_\alpha\}$ *is pro-isomorphic to* $\{0\}$.

2.6. *Corollary: If* X *and* Y *are compact spaces and*

$F : X \to Y$ *is a shape morphism, then* $F_* : \text{pro-}H_n(X) \to \text{pro-}H_n(Y)$ *is an isomorphism iff the following is true:*

(1) $F_* : \check{H}_n(X) \to \check{H}_n(Y)$ *is an isomorphism, and*

(2) $F_*^1 : \varprojlim^1 \text{pro-}H_n(X) \to \varprojlim^1 \text{pro-}H_n(Y)$ *is an isomorphism.*

The theorems in Section 3 show how to compute $F_*^1 : \varprojlim^1 \text{pro-}H_n(X) \to \varprojlim^1 \text{pro-}H_n(Y)$. The next theorem and its corollaries are analogs of 2.4, 2.5, and 2.6, but the techniques of proof are quite different.

2.7. *Theorem: Let* $\{H_i\}$ *and* $\{G_i\}$ *be inverse sequences of countable abelian groups. Let* $F : \{H_i\} \to \{G_i\}$ *be a pro-homomorphism. Then if* (1) $F_* : \varprojlim \{H_i\} \to \varprojlim \{G_i\}$ *is an isomorphism and* (2) $F_*^1 : \varprojlim^1 \{H_i\} \to \varprojlim^1 \{G_i\}$ *is an isomorphism, then* F *is an isomorphism in* pro-Ab.

2.8. *Corollary: If* $\{H_i\}$ *is an inverse sequence of countable abelian groups, then* $\{H_i\}$ *is pro-isomorphic to* $\{0\}$ *if* $\varprojlim \{H_i\} = 0$ *and* $\varprojlim^1 \{H_i\} = 0$.

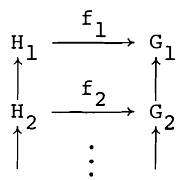
2.9. *Corollary: Let* (X,x) *and* (Y,y) *be pointed metric continua. If* $F : (X,x) \to (Y,y)$ *is a pointed shape morphism, then for* $n \geq 2$ $F_* : \text{pro-}\pi_n(X,x) \to \text{pro-}\pi_n(Y,y)$ *is an isomorphism in* pro-Ab *iff*

(1) $F_* : \check{\pi}_n(X,x) \to \check{\pi}_n(Y,y)$ *is an isomorphism, and*

(2) $F_*^1 : \varprojlim^1 \text{pro-}\pi_n(X,x) \to \varprojlim^1 \text{pro-}\pi_n(Y,y)$ *is an isomorphism.*

We conclude with an example which shows that the hypotheses of Theorems 2.4 and 2.7 cannot be relaxed very much and the results here are probably close to optimal.

2.10. Example: We will construct an example of a pro-homomorphism $F : \{H_i\} \to \{G_i\}$ which induces isomorphisms from $\varprojlim^k \{H_i\}$ to $\varprojlim^k \{G_i\}$ for all $k \geq 0$, but such that F is not a pro-isomorphism between $\{H_i\}$ and $\{G_i\}$. The systems $\{H_i\}$ and $\{G_i\}$ will be inverse sequences of abelian groups. Let $H_i = \prod_{j=1}^{\infty} Z$ for every i with the bonding homomorphisms all the identity on $\prod_{j=1}^{\infty} Z$. Let $G_i = \prod_{j=1}^{i} Z$. Let the bonding homomorphisms ξ_{ik} : $G_k \to G_i$ be the projection onto the first i coordinates. Then let $F = [f]$ where $f = (f,\{f_i\})$ is the following map of systems. The function $f : N \to N$ is the identity and $f_i : H_i = \prod_{j=1}^{\infty} Z \to G_i = \prod_{j=1}^{i} Z$ is the projection onto the first i coordinates. Then the following ladder commutes.



The induced homomorphism $F_* : \varprojlim \{H_i\} \to \varprojlim \{G_i\}$ can easily be seen to be equivalent to the identity from $\prod_{j=1}^{\infty} Z$ to $\prod_{j=1}^{\infty} Z$. However, since both systems are Mittag-Leffler and the indexing set is countable, $\varprojlim^k \{H_i\} = \varprojlim^k \{G_i\} = 0$ for all $k \geq 1$. Thus F induces an isomorphism from $\varprojlim^k \{H_i\}$ to $\varprojlim^k \{G_i\}$ for all $k \geq 0$. However, F is not a pro-isomorphism.

3. Computing Derived Limits

Since lim pro-$H_n(X)$ is just the Čech homology $\check{H}_n(X)$ which has been studied, the prinicpal difficulty in applying Corollary 2.5 is in computing $\varprojlim^1$ pro-$H_n(X)$ and the induced homomorphism $F_*^1 : \varprojlim^1$ pro-$H_n(X) \to \varprojlim^1$ pro-$H_n(Y)$. The next result gives a practical solution of this problem.

3.1. *Theorem: Let* X *be a compact space. Then* $\varprojlim^1$ pro-$H_n(X)$ *is functorially equivalent to* Ext $(\check{H}^n(X)/\mathrm{Tor}\ \check{H}^n(X), Z)$.

The Čech cohomology groups are easier to compute since direct limits of abelian groups are easier to work with than inverse limits. Also Ext(M,Z) for M an abelian group has been studied extensively (see [1] Chapter IX). Theorem 3.1 is related to Theorem 2.5 and Proposition 7.10 of [4].

There is also a way to visualize $\varprojlim^1$ of an inverse system of finitely-generated torsion free abelian groups. The next theorem is really not as specialized as it might seem. We will subsequently show that this result is a very general way to visualize $\varprojlim^1$ qualitatively.

3.2. *Theorem: Let* A *be a compact connected abelian topological group. Then* $\varprojlim^1$ pro-H_1(A) $\simeq$ A/A_0 *where* A_0 *is the arc-component of* 0 *in* A.

3.3. *Corollary: Let* A *be a compact connected abelian topological group. Then the following are equivalent.*

(1) A *is arcwise connected.*

(2) $\varprojlim^1$ pro-H_1(A) = 0.

(3) Ext($\check{H}^1$(A),Z) = 0.

(4) Ext(Char A,Z) = 0.

Corollary 3.3 shows that this result is related to the Whitehead problem in the theory of infinite abelian groups. It was an open problem until recently whether every torsion free abelian group W with Ext(W,Z) = 0 must be free abelian. S. Shelah [11] has shown that it is consistent with the usual axioms of set theory to assume that there exist groups W which are not free abelian which have Ext(W,Z) = 0. If W is countable, then it is well known that if Ext(W,Z) = 0, then W must be free abelian. Thus we have the following result for A metrizable.

3.4. *Corollary: Let* A *be a metrizable compact connected abelian topological group. Then the following are equivalent.*

(1) A *is movable.*

(2) $\varprojlim^1$ pro-H_1(A) = 0.

(3) A *is arcwise connected.*

(4) A *is* $\{0\}$, T^n, *or* T^ω.

There has been some interest in uniform movability. This is a stronger invariant than movability for nonmetric compacta. It turns out to be the same as movability for metrizable compacta. The next result shows that uniform movability is not strong enough to guarantee that $\varprojlim^1$ pro-$H_n(X) = 0$ for X a nonmetrizable compactum. Movability is enough to guarantee this if X is metrizable.

3.5. Example: Let $H = \prod_{j=1}^{\infty} Z$. Let A = Char H. In [6] the author has shown that A is movable. In [12] T. Watanabe has shown the stronger result that A is uniformly movable. It is well known that $\mathrm{Ext}(H,Z) \neq 0$ and thus A is an example of a nonmetric continuum which is movable and uniformly movable but which has $\varprojlim^1$ pro-$H_1(A) \neq 0$.

We show now that Theorem 3.2 is not as specialized as it might appear at first. It can be used to visualize the behavior of $\varprojlim^1 \{H_\alpha\}$ whenever $\{H_\alpha\}$ is any inverse system of finitely-generated abelian groups.

3.6. *Theorem: Let* $\{H_\alpha\}$ *be an inverse system of finitely-generated free abelian groups. Then there is a compact connected abelian topological group* A *such that* pro-$H_1(A)$ *is pro-isomorphic to* $\{H_\alpha\}$.

As a matter of fact let $\{G_\beta\}$ be another inverse system of finitely-generated free abelian groups and $F : \{H_\alpha\} \to \{G_\beta\}$ be a pro-homomorphism. Then if B is the compact connected abelian topological group associated with $\{G_\beta\}$, then there is a continuous homomorphism f from A to B such that $f_* :$ pro-$H_1(A) \to$ pro-$H_1(B)$ is equivalent to $F : \{H_\alpha\} \to \{G_\beta\}$. Since f is a continuous homomorphism, $f(A_0) \subset B_0$ and f induces a homomorphism from A/A_0 to B/B_0. This can be shown to be equivalent to the homomorphism induced by F from $\varprojlim^1 \{H_\alpha\}$ to $\varprojlim^1 \{G_\beta\}$. Thus one can use Theorem 3.2 and Theorem 3.6 to visualize induced homomorphisms on $\varprojlim^1$ as well.

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University of Florida

Gainesville, Florida 32611