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by

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SURVEY OF HYPERSPACE RESULTS USING INFINITE-DIMENSIONAL TOPOLOGY AND A SHORT PROOF THAT $2^I \approx Q$

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1. Introduction

Since 1972 there has been a series of results in hyperspaces whose proofs have utilized and motivated many theorems from infinite-dimensional (ID) topology. We will give a summary of the hyperspace results, as well as the corresponding ID theorems and point out their interconnections. A recent result of T. A. Chapman characterizing near-homeomorphisms on the Hilbert cube and the solution of the AR Problem by R. D. Edwards imply several of these ID theorems and consequently by applying the full power of these new results, many of the tedious verifications in the $2^I \approx Q$ (I is the closed unit interval [0,1] and Q is the Hilbert cube) and $2^{\text{graph}} \approx Q$ proofs can be eliminated; leaving the main structure of the proofs intact. In Section 4 we give a write-up of the $2^I \approx Q$ proof using these new results.

For a metric space X, let 2^X denote the space of all non-empty compact subsets of X metrized with the Hausdorff metric, and let C(X) denote the subspace of 2^X consisting of all connected elements of 2^X. In 1938, Wojdyslawski [20] proved that if X is any Peano space (i.e., compact, connected, and locally connected metric space) then 2^X is contractible and locally contractible and later [21] that 2^X is an absolute retract if and only if X is a Peano space. In his earlier paper he specifically asked if 2^I is homeomorphic to ($\approx$) the Hilbert cube Q, and, more generally, if $2^X \approx Q$ where X is any non-degenerate Peano space. Classical point-set topology techniques yielded very little when applied to this problem and it remained virtually

intact until infinite-dimensional topology played a major role in its final solution. Interestingly enough, several of the now standard theorems and techniques of infinite-dimensional topology were motivated by this and other hyperspace problems.

2. Historical Review of Hyperspace and ID Topology

The first of the recent theorems involving hyperspaces and infinite-dimensional topology was the following.

H1. (West [19]) If D *is a dendron, then* $C(D) \approx Q$ *if and only if the branch points of* D *are dense.*

West already had proved that a contractible polyhedron X is a Q-factor, that is, $X \times Q \approx Q$ (ID 4: these references refer to the theorems listed in Section 3 of this paper) and that the countable infinite product of non-degenerate Q-factors is a Hilbert cube (ID 2). Both of these theorems were used in the proof of H1, as well as two ID theorems that were motivated by these hyperspace problems, namely the Mapping Cylinder Theorem (ID 6) and the Compactification Theorem (ID 9). West also used Brown's inverse limit approximation theorem (ID 11).

The next theorem had been conjectured in 1938 by Wojdyslawski [20].

H2. (Schori and West [15]) $2^I \approx Q$.

The proof of H2 used all of the above mentioned ID theorems except the Compactification Theorem and, in addition, used an inverse limit interior approximation technique. The Attaching Theorem (ID 7) was introduced to construct Q-factors, and Q-factor decompositions and the corresponding near-homeomorphism theorem (ID 10) were introduced to establish that certain maps were near-homeomorphisms. As we see in Section 4 the use of the new results by Chapman (ID 5) and Edwards (ID 8) substantially

shortens the original proof.

H3. (Schori and West [16]) $2^{\Gamma} \approx Q$, *where* Γ *is a non-degenerate, finite, connected graph.*

The above theorem used in addition to the $2^{I} \approx Q$ result, the First Sum Theorem (ID 3), the Attaching Theorem (ID 7), and the Compactification Theorem (ID 9). In the same paper [16] it was also proved that if D is any non-degenerate local dendron, then $2^{D} \approx Q$, C(D) is a Q-factor, and if the branch points of D are dense, then $C(D) \approx Q$. These results used an additional inverse limit construction and the full strength of the Mapping Cylinder Theorem (ID 6), i.e., the part that says that the projection map stabilizes to a near-homeomorphism.

H4. (Curtis and Schori [8]) $2^{X} \approx Q$ *if and only if* X *is a non-degenerate Peano space and* $C(X) \times Q \approx Q$ *if and only if* X *is a Peano space and* $C(X) \approx Q$ *if and only if* X *is a non-degenerate Peano space containing no free arc.*

The proof uses Theorem H3 in conjunction with a delicate inverse limit construction. In the case X is a polyhedron [7], a sequence $\{\kappa_i\}$ of subdivisions of X is inductively constructed with each κ_{i+1} a subdivision of κ_i and mesh $\kappa_i \to 0$, and maps $f_i\colon 2^{\Gamma_{i+1}} \to 2^{\Gamma_i}$, where each Γ_i is the 1-skeleton of κ_i such that the limit of the inverse sequence $2^{\Gamma_1} \xleftarrow{f_1} 2^{\Gamma_2} \xleftarrow{f_2} \cdots$ is homeomorphic to 2^{X}. Each space 2^{Γ_i} is a Hilbert cube by H3 and the f_i's were constructed so that inverse images of points are contractible which by Chapman's theorem (ID 5) implies that each f_i is a near-homeomorphism. Thus, by Brown's theorem (ID 11) 2^{X} is a Hilbert cube. In the general case of X being a Peano space, a careful partitioning procedure of X is used, embedding trees in the partition members to construct the corresponding Γ_i. The proofs for the C(X) results are adapted from the 2^{X} case.

For a metric space X and $A, A_1, \cdots, A_n \in 2^X$, let $2^X_A = \{B \in 2^X: A \subset B\}$ and $2^X(A_1, \cdots, A_n) = \{B \in 2^X: A_i \cap B \neq \emptyset$ for $1 \leq i \leq n\}$.

H5. (Curtis and Schori [8] and [9]) If X is a Peano space and $A \subsetneq X$, *then* $2^X_A \approx Q$, *and if X is a non-degenerate Peano space, then* $2^X(A_1, \cdots, A_n) \approx Q$.

The first half of the theorem is proved in [8] and follows quite naturally from the proof of H4. The proof of the second part [9] required an added inverse limit interior approximation technique. Similar definitions and results exist for $C_A(X)$ and $C(X; A_1, \cdots, A_n)$.

If $\mathfrak{D}$ is a closed cover of X, let $2^X_{\mathfrak{D}} = \{A \in 2^X$: for some $D \in \mathfrak{D}$, $A \subset D\}$. Let κ be a non-degenerate finite connected complex and for $i \geq 1$, let $\kappa^{(i)}$ be the i-th barycentric subdivision of κ and let κ^0 be the 0-skeleton of κ. Then

$$Stt = St(St(\kappa^0, \kappa^{(1)}), \kappa^{(2)})$$

is a closed cover of $|\kappa|$ whose nerve is isomorphic to κ and whose interiors cover $|\kappa|$.

H6. (Curtis and Schori [9]) For non-degenerate, finite connected complexes κ, L,

(a) $2^\kappa_{Stt} \approx \kappa \times Q$,

(b) κ *and* L *have the same simple homotopy type if and only if* $2^\kappa_{Stt} \approx 2^L_{Stt}$.

The proof of part (a) uses H5 and part (b) follows directly from part (a) using Chapman's characterization of simple homotopy type [3].

If X has an affine structure, we may consider the hyperspace $cc(X) \subset 2^X$ of compact convex subsets of X.

H7. (Nadler, Quinn, Stavrakas [13])

(a) If X *is a compact convex subset of* ℓ_2 *with dim* X >1, *then* cc(X) ≈ Q.

(b) If $X \subset R^2$ *with* cc(X) ≈ Q, *then* X *must be a 2-cell.*

If $X \subset R^2$ is a 2-cell, a segment $J \subset X$ is *singular* if it contains in its interior three vertices v_1, v_2, v_3 at which X is locally non-convex, such that the side of J determined by the middle vertex v_2 is opposite that determined by v_1 and by v_3.

H8. (Curtis, Quinn, Schori [6]) If $X \subset R^2$ *is a polyhedral 2-cell, then* cc(X) ≈ Q *if and only if* X *contains no singular segments.*

A space is *continuum connected* if each pair of points in the space is contained in a subcontinuum of the space. A space is *locally continuum connected* if it has an open basis of continuum connected sets.

H9. (Curtis [5])

(a) For a metric space X, 2^X *is an* AR (metric) *if and only if* X *is connected and locally continuum connected.*

(b) $2^X \approx Q - \{pt.\}$ *if and only if* X *is a locally compact connected locally connected non-compact metric space.*

(c) A topologically complete separable connected locally connected and nowhere-locally compact metric space X *is imbeddable in a Peano continuum* P *such that* 2^X *is a pseudo-interior for* 2^P *if and only if* X *admits a metric with Property* S.

3. ID Theorems Used in Hyperspaces

A closed set A of a compact metric space X is a *z-set* in X if for each ε > 0 there is a map f: X → X - A such that d(f,id) < ε. (This ε-push definition of z-set was first used in hyperspaces.)

ID 1. Homeomorphism Extension Theorem (Anderson [1]). If

A,B *are* z-sets *in* Q *and* h: A → B *is a homeomorphism, then* h *can be extended to a homeomorphism from* Q *onto itself.*

ID2. (West [17]). If X *is a non-degenerate* Q*-factor, then* $X^{\omega} \approx Q$.

ID3. First Sum Theorem (Anderson [1]). If each of X_1, X_2 *and* $X_1 \cap X_2$ *is a* Q*-factor, and* $X_1 \cap X_2$ *is a z-set in each of* X_1 *and* X_2, *then* $X_1 \cup X_2$ *is a* Q*-factor.*

ID4. (West [17]). Each contractible polyhedron is a Q*-factor.*

By a CE-*map* we mean a continuous surjection whose inverse images of points all have trivial shape.

ID5. (Chapman [4]). CE*-maps between Hilbert cubes are near-homeomorphisms.*

The above theorems are all ID theorems that have been used extensively in hyperspace results and the following theorems and techniques with the exception of (ID 8) have the added distinction that they were motivated by work on the hyperspace problems.

ID6. Mapping Cylinder Theorem (West [18]). If X,Y *are* Q*-factors and* f: X → Y *is a map, then the mapping cylinder of* f, M(f) *is a* Q*-factor and if* c: M(f) → Y *is the projection map, then* $c \times id: M(f) \times Q \to Y \times Q$ *is a near-homeomorphism.*

ID 7. Attaching Theorem [15]. If each of A, X, *and* Y *are* Q*-factors,* A *is a z-set in* X, *and* f: A → Y *is a map, then the adjunction space of* f, $X \cup_f Y$ *is a* Q*-factor.*

The following theorem was proved in a more general setting but this statement suffices here. Furthermore, this result or

the method of proof implies all of the theorems ID 3 - ID 7.

ID 8. AR Problem (R. D. Edwards [10]). Each compact metric AR *is a* Q*-factor.* (See [4] for a write-up of this result.)

ID 9. Compactification Theorem (West [19]). If X *is a* Q*-factor,* $A \subset X$ *is a* Q*-factor and a z-set in* X*, and* $X - A$ *is a* Q*-manifold, then* X *is a Hilbert cube.*

The following definition and theorem were originally introduced in the hyperspace studies [15] to identify near-homeomorphisms.

A Q*-factor decomposition* $\mathcal{D}$ of a space X is a finite cover of X by Q-factor subsets of X such that for $D_1, D_2 \in \mathcal{D}$, (i) if $D_1 \subsetneq D_2$, then D_1 is a z-set in D_2, and (ii) if $D_1 \cap D_2 \neq \emptyset$, then $D_1 \cap D_2$ is a finite union of elements of $\mathcal{D}$.

ID 10. [15]. *Let* $f: X \to Y$ *be a map such that for each* $\varepsilon > 0$ *there exists a* Q*-factor decomposition* $\mathcal{D}$ *of* Y *with mesh* $\mathcal{D} < \varepsilon$ *such that* $f^{-1}(\mathcal{D})$ *is a* Q*-factor decomposition of* X*, then* $f \times \mathrm{id}: X \times Q \to Y \times Q$ *is a near-homeomorphism.*

The following theorems are listed because they have become the main tools for identifying Hilbert cubes in hyperspace problems as well as in infinite-dimensional topology.

ID 11. (Morton Brown [2]*). Let* $S = \mathrm{inv}\lim(X_n, f_n)$ *where each* X_n *is homeomorphic to a compact metric space* X *and each* f_n *is a near-homeomorphism, then* S *is homeomorphic to* X.

ID 12. Inverse limit interior approximation theorem [7]. *For each* $n \geq 1$, *let* Y_n *be a closed subset of a compact metric space* Y *and let* $f_n: Y_{n+1} \to Y_n$ *be a map such that*

(1) $Y_n \to Y$ *(in* 2^Y*)*

(2) $\sum_{n=1}^{\infty} d(f_n, \mathrm{id}) < \infty$, *and*

(3) $\{f_i \circ \cdots \circ f_j; j \geq i\}$ *is an equi-uniformly continuous family for each* i.

Then $Y \approx \text{inv lim}(Y_n, f_n)$.

4. Short Proof that $2^I \approx Q$

The proof that $2^I \approx Q$ given here is basically the original proof where the powerful theorems of Chapman (ID 5) and Edwards (ID 8) eliminate the longest and most technical parts, that is, showing certain maps are near-homeomorphisms and certain spaces are Q-factors.

The following lemma was first observed by Fort and Segal [11] and can be thought of as a special case of (ID 12). It provides a useful method for recognizing certain inverse limits.

Lemma 4.1. [*11, Lemma 4, p. 132*]. *Let* X *be a compact metric space, let* $X_1, X_2, \cdots$ *be closed subsets of* X, *and for each* n *let* ϕ_n *be a map of* X *onto* X_n *and let* f_n *be a map of* X_{n+1} *onto* X_n *such that* $\phi_n = f_n \circ \phi_{n+1}$ *and* $\phi_1, \phi_2, \cdots$ *converges uniformly to the identity map on* X. *Then the function* ϕ *on* X *defined by* $\phi(x) = (\phi_1(x), \phi_2(x), \cdots)$ *is a homeomorphism of* X *onto* $\text{inv lim}(X_n, f_n)$.

For $S \subset I$, let $H(S)$ denote the space 2^I_S and denote $H(\{0,1\})$ by $H(0,1)$ which has often been denoted by 2^I_{01}.

The first result says that it is sufficient to prove that $H(0,1) \approx Q$.

Proposition 4.2. *If* $H(0,1)$ *is a Hilbert cube, then so is* 2^I.

Proof. In [14] it is shown that 2^I is homeomorphic to $CCH(0,1)$, where CX denotes the cone over X. (The formula $(A,s,t) \to \{(1-t)(1-s)a + t : a \in A\}$ defines a map from $H(0,1) \times I \times I$ to 2^I producing the same identifications as the coning operations.) O. H. Keller proved in [12] that all infinite-dimensional, convex compact subsets of Hilbert space

are Hilbert cubes, and since CQ has a geometric realization as such a subset of Hilbert space, then CQ and, hence, CCQ is a Hilbert cube and the result follows.

Theorem 4.3. $H(0,1)$ *is a Q-factor.*

Proof. By [21], 2^I is an AR and by showing that $H(0,1)$ is a retract of 2^I we will have that $H(0,1)$ is an AR and, hence, by (ID 8), $H(0,1)$ is a Q-factor. Define $f_t\colon 2^I \to 2^I$, $0 \leq t \leq 1$, by letting $f_t(A)$ be the closed t-neighborhood of A in I and let $\phi(A) = \inf\{t\colon f_t(A) \in H(0,1)\}$. Then $r\colon 2^I \to H(0,1)$ defined by $r(A) = f_{\phi(A)}(A)$ is a retraction.

For each $n \geq 1$, let $\sigma(n) = \{0,1,1/n,1/n+1,\cdots\}$ and let $Y_n = H(\sigma(n))$.

Corollary 4.4. Each Y_n *is a Hilbert cube.*

Proof. For a fixed $n \geq 1$, let J_m denote the m-th subinterval from the right determined by $\sigma(n)$, i.e., $J_1 = [1/n,1]$, $J_2 = [1/n+1,1/n]$, etc., and let $H_m = \{A \in 2^{J_m}\colon A$ contains the end-points of $J_m\}$. Then $\alpha\colon Y_n \to \prod_{m=1}^{\infty} H_m$ defined by $\alpha(A) = (A \cap J_1, A \cap J_2, \cdots)$ is a homeomorphism. Since each H_m is homeomorphic to $H(0,1)$, then Y_n is topologically a countable infinite product of copies of a non-degenerate Q-factor $H(0,1)$ which by (ID 2) is a Hilbert cube.

We define maps $r_n\colon Y_{n+1} \to Y_n$ as follows. For $A \in Y_{n+1}$, let $u = \max\{x \in A\colon x \leq 1/n\}$ and $v = \min\{x \in A\colon x \geq 1/n\}$, and let $\alpha = \min\{d\colon A \cup [u,u+d] \cup [v-d,v] \in Y_n\}$. Then $r_n(A) = A \cup [u,u+\alpha] \cup [v-\alpha,v]$. Note that $\alpha = \min\{1/n - u, v - 1/n\}$. It is easy to see that each r_n is continuous and a retraction.

Proposition 4.5. The inverse limit of (Y_n,r_n) *is homeomorphic to* $H(0,1)$.

Proof. For $x \in I$ and $B \subset I$, let $d(x,B) = \inf\{|x - b|\colon b \in B\}$ and for $U \subset I$ and $n \geq 1$, let $\xi(U,n) = \max\{d(x,I - U)\colon x \in \sigma(n)\}$.

Define R_n: $H(0,1) \to Y_n$ by letting $R_n(A)$ be the union of A and $\bigcup\{[u,u + \xi(U,n)] \cup [v - \xi(U,n),v]: U = (u,v)$ is a component of $I - A\}$. It easily follows that $R_n = r_n \circ R_{n+1}$ by observing what happens, for $A \in H(0,1)$, on the component of $I - A$ that contains $1/n$, if it exists. Thus, we can define R: $H(0,1) \to \text{inv lim}(Y_n,r_n)$ by $R(A) = (R_1(A),R_2(A),\cdots)$ and this is a homeomorphism by 4.1 since $R_1,R_2,\cdots$ converges uniformly to the identity map on $H(0,1)$.

Theorem 4.6. $2^I \approx Q$.

Proof. By (ID 11) and the above results all we need do is verify that each r_n: $Y_{n+1} \to Y_n$ is a near-homeomorphism and this is reduced by (ID 5) to showing that each r_n is a CE-map. In fact, not only is the inverse image under r_n of each point of Y_n of trivial shape, but it is contractible. To see, this, using the notation introduced for defining r_n, we define h_t: $Y_{n+1} \to Y_{n+1}$ by $h_t(A) = A \cup [u,u + t\alpha] \cup [v - t\alpha,v]$ obtaining a homotopy such that $h_0 = \text{id}$, $h_1 = r_n$, and $r_n h_t(A) = r_n(A)$ for each $t \in I$ and $A \in Y_{n+1}$. Thus, for $B \in Y_n$, $h_t | r_n^{-1}(B)$ is a contraction of $r_n^{-1}(B)$ to B since if $A \in r_n^{-1}(B)$, then $r_n h_t(A) = r_n(A)$ implies that $h_t(A) \in r_n^{-1}(B)$. Thus, point inverses under r_n are contractible and hence r_n is a near-homeomorphism and the proof is complete.

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