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In [15] Whitney proved that for any continuum X there exists a map μ from C(X), the hyperspace of subcontinua of X, into $[0,\infty)$, satisfying (1) $\mu(\{x\}) = 0$ for all $x \in X$ and (2) if $A \subsetneq B$, then $\mu(A) < \mu(B)$. Any map from C(X) into $[0,\infty)$ satisfying (1) and (2) is called a Whitney map. Point inverses $\mu^{-1}(t)$ for $t \in [0,\mu(X)]$ are continua [2]; they are called Whitney continua. A topological property P is called a Whitney property if whenever X has property P, so does every Whitney continuum in C(X).

Since Kelley introduced Whitney maps into the study of hyperspaces in 1942 [6], these maps have played an important role in the investigation of the hyperspace C(X). Recently Krasinkiewicz [8], Nadler [8, 10] and Rogers [12, 13, 14] have studied Whitney continua and have determined whether or not certain topologocal properties are Whitney properties. This paper is closely related to the work of these three authors, and many of the results have been obtained in answer to questions raised by them.

In Section 1 we deal with the question of whether the possession of a Whitney property P by $\mu^{-1}(t_0)$ implies that $\mu^{-1}(t)$ also has the property P for all $t \in (t_0,\mu(X)]$. This is true for local connectivity, arcwise connectivity and hereditary indecomposability. We obtain a partial answer to the question for decomposability. In the second section we show that aposyndesis, finite aposyndesis, mutual aposyndesis and semi-aposyndesis are Whitney properties.

Kelley has shown [6] that the function σ defined on C(C(X)) by $\sigma\mathcal{A} = \cup\{A | A \in \mathcal{A}\}$ is a continuous function from C(C(X)) onto

$C(X)$. We make use of the map σ in Section 3, where we investigate continua having the following properties: (1) for any μ, any $t \in [0,\mu(X)]$ and any subcontinuum $\mathfrak{A}$ of $\mu^{-1}(t)$, if $\sigma\mathfrak{A} = X$, then $\mathfrak{A} = \mu^{-1}(t)$; (2) for any μ, any $t \in [0,\mu(X)]$ and any subcontinuum $\mathfrak{A}$ of $\mu^{-1}(t)$, if $A \in \mu^{-1}(t)$ and $A \subseteq \sigma\mathfrak{A}$ then $A \in \mathfrak{A}$. It has been shown [8] that arc-like and hereditarily indecomposable continua have property (1). We show that non-planar circle-like continua have property (2). All continua having property (2) are shown to be atriodic. Moreover, if X has property (2), we prove that the "top of the hyperspace" is a hyperspace, i.e. we prove that for any $t \in [0,\mu(X)]$, $\mu^{-1}([t,\mu(X)])$ is homeomorphic in a natural way to $C(\mu^{-1}(t))$. The homeomorphism induces a Whitney map on $C(\mu^{-1}(t))$ which enables us to draw some conclusions about Whitney properties in $C(X)$.

In Section 4 we answer negatively Nadler's question [10] of whether the topological types of Whitney map inverses is a topological invariant.

A continuum is a non-empty compact connected metric space. The letter X will always denote a continuum, and d will be a metric on X. For $x \in X$ and $\varepsilon > 0$, let $B_d(x;\varepsilon) = \{y \in X \mid d(x,y) < \varepsilon\}$. If $A \subseteq X$, then let $V_\varepsilon(A) = \bigcup\{B_d(a;\varepsilon) \mid a \in A\}$. The closure of A and the interior of A will be denoted by $\overline{A}$ and int A respectively. A map is a continuous function.

Let ρ be the Hausdorff metric on $C(X)$ [6]. The symbol μ represents a Whitney map on $C(X)$. Let $\hat{X}$ be the subset of $C(X)$ consisting of all singleton subsets of X. If $A \subseteq X$, then let $C(A) = \{Y \in C(X) \mid Y \subseteq A\}$. For a fixed μ and for $t \in [0,\mu(X)]$, let $C(A,t) = \{Y \in \mu^{-1}(t) \mid Y \subseteq A\}$ and $C_A^t = \{Y \in \mu^{-1}(t) \mid A \subseteq Y\}$. If $A = \{p\}$, write C_p^t for C_A^t. If $A \in C(X)$, then $C(A,t)$ is a continuum, since it is a Whitney continuum in $C(A)$. A simple extension of Theorem 4.2 of [14] shows that C_A^t is a continuum when $A \in C(X)$.

In his classical work on hyperspaces [6], J. L. Kelley defined a segment in C(X) as follows: if $A,B \in C(X)$ and $A \subsetneq B$, then a segment from A to B is an arc $\{A_t | t \in [0,1]\}$ in C(X) such that (1) $A_0 = A_1$, (2) $A_1 = B$, (3) $\mu(A_t) = (1-t)\mu(A) + t\mu(B)$, and (4) if $t' < t''$, then $A_{t'} \subsetneq A_{t''}$. Kelley [6] proved that if $A,B \in C(X)$ and $A \subsetneq B$, then there exists a segment from A to B.

Nadler [10] and Rogers [14] obtained the following result:

Lemma 0. For continua $A,B \in C(X)$, *if* $\mu(A) = \mu(B) = t$, $A \cap B \neq \emptyset$, *and* C *is a component of* $A \cap B$, *then there exists an arc in* $\mu^{-1}(t)$ *from* A *to* B *such that each point of the arc is a continuum which contains* C *and is contained in* $A \cup B$.

Another very useful fact was proved by Krasinkiewicz in [7]:

Lemma 00. If $\mu(A) = \mu(B)$ *and* $\varepsilon > 0$, *then there exists* $\eta > 0$ *such that* $B \subseteq V_\eta(A)$ *implies* $\rho(A,B) < \varepsilon$.

We will make frequent use of these two lemmas in the remainder of this paper.

1. Relationships Among Whitney Continua

In considering Whitney properties we deal with the relationship between $\mu^{-1}(0)$ and $\mu^{-1}(t)$ for $t \in (0,\mu(X)]$. In this section we extend this idea and consider the relationship between $\mu^{-1}(t_1)$ and $\mu^{-1}(t_2)$ for $t_1, t_2 \in [0,\mu(X)]$. We provide some answers to the following question raised by Nadler in [10]: If $\mu^{-1}(t_0)$ has a Whitney property P and $t \in (t_0,\mu(X)]$, does $\mu^{-1}(t)$ have the same property?

We note that Nadler [10] has shown that local connectivity is a Whitney property and that Nadler [10] and Rogers [14] proved that arcwise connectivity is a Whitney property. Propositions 1 and 2 will extend these results.

Proposition 1. If $\mu^{-1}(t_0)$ *is locally connected, then* $\mu^{-1}(t)$ *is locally connected for* $t \in (t_0, \mu(X)]$.

Proof. Let $0 \le t_0 < t \le \mu(X)$, $A \in \mu^{-1}(t)$ and let $B_\rho(A;\varepsilon) \cap \mu^{-1}(t)$ be a neighborhood of A in $\mu^{-1}(t)$. For $\mathfrak{A} \subseteq C(X)$ and $s \in [0, \mu(X)]$, let $C(\mathfrak{A}, s)$ denote $\bigcup \{C(M;s) \mid M \in \mathfrak{A}\}$.

Step 1: Show that $C(B_\rho(A;\varepsilon), t_0)$ is a neighborhood of $C(A,t_0)$ in $\mu^{-1}(t_0)$.

Since $\mu^{-1}(t_0)$ is locally connected, it has Kelley's Property 3.2 [6], i.e. there exists $\delta = \delta(\varepsilon)$ such that if $Y_1 \in \mathcal{Y}_1 \in C(\mu^{-1}(t_0))$ and $Y_2 \in B_\rho(Y_1;\delta) \cap \mu^{-1}(t_0)$, then there exists $\mathcal{Y}_2 \in C(\mu^{-1}(t_0))$ such that $Y_2 \in \mathcal{Y}_2$ and $\rho^2(\mathcal{Y}_1, \mathcal{Y}_2) < \varepsilon$ (where ρ^2 is the Hausdorff metric on $C(C(X))$).

Let $X_2 \in V_\delta(C(A,t_0)) \cap \mu^{-1}(t_0)$. Then there exists $X_1 \in C(A,t_0)$ such that $X_2 \in B_\rho(X_1;\delta) \cap \mu^{-1}(t_0)$. So by the choice of δ, there exists $\mathfrak{A} \in C(\mu^{-1}(t_0))$ with $X_2 \in \mathfrak{A}$ and $\rho^2(C(A,t_0), \mathfrak{A}) < \varepsilon$. Now $\sigma\colon C(C(X)) \to C(X)$ is a distance-reducing function [1.1 of 6], so $\rho(\sigma C(A,t_0), \sigma\mathfrak{A}) < \varepsilon$. Since $\sigma C(A,t_0) = A$, we have $\rho(A, \sigma\mathfrak{A}) < \varepsilon$. Thus $\sigma\mathfrak{A} \in B_\rho(A;\varepsilon)$ and so $X_2 \in C(\sigma\mathfrak{A}, t_0) \subseteq C(B_\rho(A;\varepsilon), t_0)$. Hence $V_\delta(C(A,t_0)) \cap \mu^{-1}(t_0) \subseteq C(B_\rho(A;\varepsilon), t_0)$, and so $C(B_\rho(A;\varepsilon), t_0)$ is a neighborhood of $C(A,t_0)$ in $\mu^{-1}(t_0)$.

Step 2: Obtain a continuum $\overline{\mathcal{U}}$ which is a neighborhood of $C(A,t_0)$ in $\mu^{-1}(t_0)$ and is contained in $C(B_\rho(A;\varepsilon), t_0)$.

By Lemma 00 there exists η such that if $A, B \in \mu^{-1}(t)$ and $B \subseteq V_\eta(A)$, then $\rho(A,B) < \varepsilon$. Choose $\gamma = \min\{\eta, \delta\}$. Since $\mu^{-1}(t_0)$ is a locally connected continuum and $C(A,t_0)$ is a continuum, there exists a connected neighborhood $\mathcal{U}$ of $C(A,t_0)$ in $\mu^{-1}(t_0)$ such that $\mathcal{U} \subseteq V_{\frac{1}{2}\gamma}(C(A,t_0)) \cap \mu^{-1}(t_0)$. Then $\overline{\mathcal{U}}$ is a continuum and $\overline{\mathcal{U}} \subseteq V_\gamma(C(A,t_0)) \cap \mu^{-1}(t_0) \subseteq V_\delta(C(A,t_0)) \cap \mu^{-1}(t_0) \subseteq C(B_\rho(A;\varepsilon), t_0)$.

Now $\sigma\overline{\mathcal{U}}$ is a continuum and $C(A,t_0) \subseteq \operatorname{int} \overline{\mathcal{U}} \subseteq \operatorname{int} C(\sigma\overline{\mathcal{U}}, t_0)$.

Step 3: Show that $C(\sigma\overline{\mathcal{U}}, t) \subseteq B_\rho(A;\varepsilon) \cap \mu^{-1}(t)$.

Let $B \in C(\sigma\overline{\mathcal{U}}, t)$. Then $B \subseteq \sigma\overline{\mathcal{U}}$, so for every $b \in B$ there exists $B_b \in \overline{\mathcal{U}}$ such that $b \in B_b$. But $\overline{\mathcal{U}} \subseteq V_\gamma(C(A,t_0))$, so there exists $A_b \in C(A,t_0)$ such that $\rho(B_b,A_b) < \gamma$. Thus $b \in V_\gamma(A_b)$ and it follows that $B \subseteq V_\gamma(A) \subseteq V_\eta(A)$. Then, by the choice of η, $\rho(A,B) < \varepsilon$. So $B \in B_\rho(A;\varepsilon)$ and hence $C(\sigma\overline{\mathfrak{U}},t) \subseteq B_\rho(A;\varepsilon) \cap \mu^{-1}(t)$.

Step 4: Since $C(\sigma\overline{\mathfrak{U}},t)$ is a continuum, we may complete the proof by showing that $A \in \text{int}\, C(\sigma\overline{\mathfrak{U}},t)$.

Suppose $A \notin \text{int}\, C(\sigma\overline{\mathfrak{U}},t)$. Then there exists a sequence $\{X_n\}$ of elements of $\mu^{-1}(t)\setminus C(\sigma\overline{\mathfrak{U}},t)$ such that $X_n \to A$. Since $X_n \not\subseteq \sigma\overline{\mathfrak{U}}$, for each n there exists $x_n \in X_n\setminus\sigma\overline{\mathfrak{U}}$ and $E_n \in \mu^{-1}(t_0)$ such that $x_n \in E_n \subseteq X_n$. Now some subsequence of $\{E_n\}$ converges, so suppose $E_n \to E$. Then $E \in \mu^{-1}(t_0)$. For any $y \in E$ there is a sequence $\{y_n\}$, $y_n \in E_n$, with $y_n \to y$. But $y_n \in E_n \subseteq X_n$ and $X_n \to A$, so $y \in A$ and hence $E \in C(A,t_0)$. Since $\overline{\mathfrak{U}}$ is a neighborhood of $C(A,t_0)$ in $\mu^{-1}(t_0)$, there exists N such that $n > N$ implies $E_n \in \overline{\mathfrak{U}}$ and hence $x_n \in \sigma\overline{\mathfrak{U}}$. This contradicts the assumption that $x_n \in X_n\setminus\sigma\overline{\mathfrak{U}}$. Therefore, $A \in \text{int}\, C(\sigma\overline{\mathfrak{U}},t)$.

The proof of the next proposition is based on the idea in the proofs of Theorem 3.8 of [14] and Theorem 2 of [10].

Proposition 2. If $\mu^{-1}(t_0)$ *is arc-connected, then* $\mu^{-1}(t)$ *is arc-connected for* $t \in (t_0,\mu(X)]$.

Proof. Let $0 \le t_0 < t \le \mu(X)$ and $A,B \in \mu^{-1}(t)$. Let $A' \in C(A,t_0)$ and $B' \in C(B,t_0)$. Since $\mu^{-1}(t_0)$ is arc-connected, there is an arc γ in $\mu^{-1}(t_0)$ from A' to B'. We consider two cases.

Case I: Suppose $\mu(\sigma(\gamma)) \le t$.

Let $\phi\colon [0,1] \to C(B)$ be a segment from B' to B. Since $\mu(\sigma(\gamma) \cup B') = \mu(\sigma(\gamma)) \le t$ and $\mu(\sigma(\gamma) \cup B) \ge \mu(B) = t$, there exists $u_0 \in [0,1]$ such that $\mu(\sigma(\gamma) \cup \phi(u_0)) = t$. Now $\mu(A) = t$ and $A \cap (\sigma(\gamma) \cup \phi(u_0)) \supseteq A' \neq \emptyset$, so by Lemma 0 there is an arc in $\mu^{-1}(t)$ from A to $\sigma(\gamma) \cup \phi(u_0)$. Similarly, there is an arc in $\mu^{-1}(t)$ from $\sigma(\gamma) \cup \phi(u_0)$ to B. Hence there is an arc in $\mu^{-1}(t)$ from A to B.

Case II: Suppose $\mu(\sigma(\gamma)) > t$.

By the uniform continuity of μ there exists δ such that if $R,S \in C(X)$ and $\rho(R,S) < \delta$, then $|\mu(R) - \mu(S)| < t - t_0$. Now γ is locally connected, so for each $Y \in \gamma$, the neighborhood $B_\rho(Y;\frac{1}{2}\delta) \cap \gamma$ of Y in γ contains a connected neighborhood $U(Y)$ of Y in γ. Some finite collection $\{U(Y_1),\cdots,U(Y_n)\}$, for $Y_1,\cdots,Y_n \in \gamma$, covers γ.

For each i, $\overline{U(Y_i)}$ is a subinterval $[A_i,B_i]$ of γ. Let $\{C_i | i=1,\cdots,2n\} = \{A_i,B_i | i=1,\cdots,n\}$ with $0 = \gamma^{-1}(A') = \gamma^{-1}(C_1) \leq \gamma^{-1}(C_2) \leq \cdots \leq \gamma^{-1}(C_{2n}) = \gamma^{-1}(B') = 1$. Then each interval $[C_i,C_{i+1}]$, $i=1,\cdots,2n-1$, is contained in some $\overline{U(Y_{j(i)})} \subseteq \overline{B_\rho(Y_{j(i)};\frac{1}{2}\delta)}$. So $\rho(Y_{j(i)},\sigma([C_i,C_{i+1}])) < \delta$ for all i. This implies, by the choice of δ, that $t - t_0 > |\mu(\sigma([C_i,C_{i+1}])) - \mu(Y_{j(i)})| = |\mu(\sigma([C_i,C_{i+1}])) - t_0|$, and thus $\mu(\sigma([C_i,C_{i+1}])) < t$. Since $\mu(\sigma(\gamma)) > t$, there exists a continuum $\mathcal{D}_i$ such that $[C_i,C_{i+1}] \subseteq \mathcal{D}_i \subseteq \gamma$ and $\mu(\sigma(\mathcal{D}_i)) = t$.

So for each $i=1,\cdots,2n-1$, we have $\sigma(\mathcal{D}_i) \in \mu^{-1}(t)$ and $C_{i+1} \subseteq \sigma(\mathcal{D}_i) \cap \sigma(\mathcal{D}_{i+1}) \neq \emptyset$. By Lemma 0 there exist arcs in $\mu^{-1}(t)$ from $\sigma(\mathcal{D}_i)$ to $\sigma(\mathcal{D}_{i+1})$. Since $A' \subseteq A \cap \sigma(\mathcal{D}_1)$ and $B' \subseteq B \cap \sigma(\mathcal{D}_{2n-1})$, there are arcs in $\mu^{-1}(t)$ from A to $\sigma(\mathcal{D}_1)$ and from $\sigma(\mathcal{D}_{2n-1})$ to B. Therefore there is an arc in $\mu^{-1}(t)$ from A to B.

Corollary 3. If $\mu^{-1}(t_0)$ *has* n *arc components,* $2 \leq n < \infty$ *and* $t \in (t_0,\mu(X)]$, *then* $\mu^{-1}(t)$ *has at most* n *arc components.*

Proof. Suppose A and B are in different arc components of $\mu^{-1}(t)$. If there were an arc in $\mu^{-1}(t_0)$ from any point of $C(A,t_0)$ to a point of $C(B,t_0)$, then by the proof of Proposition 2, there would be an arc in $\mu^{-1}(t)$ from A to B. So there is at least one arc component in $\mu^{-1}(t_0)$ for each arc component in $\mu^{-1}(t)$.

Corollary 4. If X *is decomposable, then there exists*

$t_0 \in [0,\mu(X))$ *such that* $\mu^{-1}(t)$ *is non arc-connected if* $t < t_0$ *and arc-connected if* $t > t_0$.

Proof. If X is arc-connected, take $t_0 = 0$. If X is not arc-connected, then by Theorem 3.5 of [8] there exists t' such that $\mu^{-1}(t)$ is arc-connected for $t \geq t'$. Let t_0 = g.l.b. $\{t' | \mu^{-1}(t)$ is arc-connected for all $t \geq t'\}$. Then $\mu^{-1}(t)$ is arc-connected for $t > t_0$. Suppose there exists $t_1 < t_0$ such that $\mu^{-1}(t_1)$ is arc-connected. Then by Proposition 2, $\mu^{-1}(t)$ is arc-connected for all $t > t_1$, contradicting the choice of t_0. Thus $\mu^{-1}(t)$ is non arc-connected for $t < t_0$.

We turn now to a consideration of the relationships among Whitney continua with regard to decomposability properties. We need the following lemma, which is a generalization of Corollary 3.2 of [8].

Lemma 5. If $\mathfrak{A} \in C(\mu^{-1}(t_0))$ *and* $t \in (t_0,\mu(X)]$ *then* $X(\mathfrak{A},t) = \cup\{C_A^t | A \in \mathfrak{A}\}$ *is a subcontinuum of* $\mu^{-1}(t)$. *If* $\mu(\sigma\mathfrak{A}) \leq t$, *then* $X(\mathfrak{A},t)$ *is arc-connected.*

Proof. The compactness of $\mathfrak{A}$ readily implies that of $X(\mathfrak{A},t)$.

Suppose $X(\mathfrak{A},t)$ is not connected. Then $X(\mathfrak{A},t) = \mathcal{C} \cup \mathfrak{D}$ where $\mathcal{C}$ and $\mathfrak{D}$ are disjoint, non-empty and open in $X(\mathfrak{A},t)$. Since $X(\mathfrak{A},t) = \cup\{C_A^t | A \in \mathfrak{A}\}$ and C_A^t is a continuum for each $A \in \mathfrak{A}$, we may write $\mathfrak{A} = \mathcal{C}' \cup \mathfrak{D}'$ where $\mathcal{C}' = \{A \in \mathfrak{A} | C_A^t \subseteq \mathcal{C}\} \neq \emptyset$, $\mathfrak{D}' = \{A \in \mathfrak{A} | C_A^t \subseteq \mathfrak{D}\} \neq \emptyset$ and $\mathcal{C}' \cap \mathfrak{D}' = \emptyset$. Since $\mathcal{C}$ and $\mathfrak{D}$ are open in $X(\mathfrak{A},t)$, it is easy to show that $\mathcal{C}'$ and $\mathfrak{D}'$ are open in $\mathfrak{A}$. This is impossible since $\mathfrak{A}$ is a continuum. So $X(\mathfrak{A},t)$ is connected and hence a subcontinuum of $\mu^{-1}(t)$.

Suppose $\mu(\sigma\mathfrak{A}) \leq t$. Let $D \in \mu^{-1}(t)$ such that $\sigma\mathfrak{A} \subseteq D$. If $B \in X(\mathfrak{A},t)$, then $B \in C_A^t$ for some $A \in \mathfrak{A}$. But $A \subseteq \sigma\mathfrak{A} \subseteq D$, so $A \subseteq B \cap D$. By Lemma 0 there is an arc in $\mu^{-1}(t)$ from B to D. Hence $X(\mathfrak{A},t)$ is arc-connected.

Proposition 6. If $\mu^{-1}(t_0)$ *is decomposable, then there exists* $t_1 \in (t_0, \mu(X)]$ *such that* $\mu^{-1}(t)$ *is decomposable for* $t_0 \le t \le t_1$.

Proof. Let $\mu^{-1}(t_0) = \mathfrak{A} \cup \mathfrak{B}$ be a decomposition. Let $B \in \mathfrak{B} \setminus \mathfrak{A}$ and $\delta = \rho(B, \mathfrak{A})$. By Lemma 00 there exists ε such that if $R, S \in \mu^{-1}(t_0)$ and $R \subseteq V_\varepsilon(S)$, then $\rho(R,S) < \delta$. There exists $\hat{B} \in C(X)$ such that $B \subsetneq \hat{B} \subseteq V_\varepsilon(B)$. Then $\hat{B}$ contains no point of $\mathfrak{A}$, since $\rho(B, \mathfrak{A}) = \delta$. Similarly, if $A \in \mathfrak{A} \setminus \mathfrak{B}$, then there exists $\hat{A} \in C(X)$ such that $A \subsetneq \hat{A}$ and $\hat{A}$ contains no point of $\mathfrak{B}$. Let $t_1 = \min\{\mu(\hat{A}), \mu(\hat{B})\}$. Let $t_0 \le t \le t_1$. Then there exist $A_t, B_t \in \mu^{-1}(t)$ such that $A \subseteq A_t \subseteq \hat{A}$ and $B \subseteq B_t \subseteq \hat{B}$. Since $\mu^{-1}(t_0) = \mathfrak{A} \cup \mathfrak{B}$, we have $\mu^{-1}(t) = X(\mathfrak{A}, t) \cup X(\mathfrak{B}, t)$. This is a decomposition since $X(\mathfrak{A}, t)$ and $X(\mathfrak{B}, t)$ are continua and $A_t \in X(\mathfrak{A}, t) \setminus X(\mathfrak{B}, t)$ and $B_t \in X(\mathfrak{B}, t) \setminus X(\mathfrak{A}, t)$. Therefore, $\mu^{-1}(t)$ is decomposable.

Although decomposability is a Whitney property [8], neither hereditary decomposability nor its opposite, non hereditary decomposability, is a Whitney property. This is shown by the following examples.

Examples 7. a) Let $X = A_1 \cup A_2 \cup A_3$ where each A_i is a copy of $I = [0,1]$ and $A_i \cap A_j = \{0\}$ for $i \ne j$. Then X is hereditarily decomposable. Let μ_1 be the Whitney map on $C(I)$ for which $\mu_1([a,b]) = b - a$ for all $[a,b] \in C(I)$. Define $\mu\colon C(X) \to [0,\infty)$ by $\mu(A) = \mu_1(A \cap A_1) + \mu_1(A \cap A_2) + \mu_1(A \cap A_3)$. Then μ is a Whitney map.

Let $t \in (0,3)$ and let $B = \{(x_1,x_2,x_3) \mid x_1 + x_2 + x_3 = t\}$. Define $f\colon B \to \mu^{-1}(t)$ by letting $f(x_1,x_2,x_3)$ be the continuum A such that $A \cap A_i = [0,x_i]$. Then B is a 2-cell and f is a homeomorphism. So $\mu^{-1}(t)$ contains a 2-cell and hence is not hereditarily decomposable.

b) Let Y be an hereditarily indecomposable continuum and let a_1, a_2 and a_3 be points of Y contained in three different composants. Attach arcs A_i, i = 1,2,3, to Y so that $A_i \cap Y = \{a_i\}$ and $A_i \cap A_j = \emptyset$ for $i \neq j$. Let $X = Y \cup \bigcup_{i=1}^{3} A_i$. Then X is not hereditarily decomposable, but $\mu^{-1}(\mu(Y))$ is homeomorphic to a figure T and so is hereditarily decomposable.

It has been shown that indecomposability is not a Whitney property [6, 14]. However, hereditary indecomposability is a Whitney property [6], and we also have the following proposition.

Proposition 8. If $\mu^{-1}(t_0)$ *is hereditarily indecomposable and* $t \in (t_0, \mu(X)]$, *then* $\mu^{-1}(t)$ *is hereditarily indecomposable.*

Proof. If there exists $A \in \mu^{-1}(t)$, for some $t \in (t_0, \mu(X)]$, such that A is decomposable, then $C(A, t_0)$ is a decomposable subcontinuum of $\mu^{-1}(t_0)$. But $\mu^{-1}(t_0)$ is hereditarily indecomposable. So if $A \in C(X)$ and A is decomposable, then $\mu(A) \leq t_0$.

Suppose $\mathfrak{A} \in C(\mu^{-1}(t))$ and $\mathfrak{A} = \mathfrak{A}_1 \cup \mathfrak{A}_2$ is a decomposition of $\mathfrak{A}$. Since $\mu(\sigma\mathfrak{A}) > t_0$, $\sigma\mathfrak{A}$ is indecomposable. So for at least one of $\mathfrak{A}_1$ and $\mathfrak{A}_2$, say $\mathfrak{A}_1$, we must have $\sigma\mathfrak{A} = \sigma\mathfrak{A}_1$. Choose $A_2 \in \mathfrak{A}_2 \setminus \mathfrak{A}_1$. Then $A_2 \subseteq \sigma\mathfrak{A} = \sigma\mathfrak{A}_1$, so there exists $A_1 \in \mathfrak{A}_1$ such that $A_1 \cap A_2 \neq \emptyset$. Then since $A_1 \neq A_2$, $A_1 \cup A_2$ is a decomposable continuum, and $\mu(A_1 \cup A_2) > t > t_0$. This is impossible, so $\mu^{-1}(t)$ is hereditarily indecomposable.

2. Aposyndesis Properties of Whitney Continua

In light of the local connectivity properties of Whitney continua discussed in Section 1, we proceed now to an examination of aposyndesis in Whitney continua. Aposyndesis is a weaker property than local connectivity.

The concept of aposyndesis was introduced by F. Burton Jones [5] in 1941. If $p, q \in X$, $p \neq q$, then X is said to be aposyndetic at p with respect to q provided there exists a continuum M such that $p \in \text{int } M$ and $q \in X \setminus M$. If X is aposyndetic at p with respect to each point $q \in X \setminus \{p\}$, then X is aposyndetic at p. We say that X is finitely aposyndetic if for any $x \in X$ and $\{x_1, \cdots, x_n\} \subseteq X$ with $x \neq x_i$, there exists a continuum M such that $x \in \text{int } M$ and $M \cap \{x_1, \cdots, x_n\} = \emptyset$. If X is aposyndetic at each of its points, then X is aposyndetic. If for every pair of points (p,q) of X, X is aposyndetic at at least one of the points with respect to the other, then X is semi-aposyndetic.

Stronger than aposyndesis is the property of mutual aposyndesis: X is said to be mutually aposyndetic if for any $p, q \in X$, $p \neq q$, there exist disjoint continua M and N with $p \in \text{int } M$, $q \in \text{int } N$. If for any countable closed set F in X and any $x \in X \setminus F$, there is a continuum M such that $x \in \text{int } M$ and $M \cap F = \emptyset$, then X is countable closed set aposyndetic. A good reference for variations on the idea of aposyndesis is [4].

J. Goodykoontz [3] has shown that C(X) is aposyndetic, in fact, countable closed set aposyndetic; moreover, if X is semi-aposyndetic, then C(X) is mutually aposyndetic. We will show that several aposyndesis properties are Whitney properties.

Proposition 9. Aposyndesis is a Whitney property.

Proof. Suppose X is aposyndetic and let $A, B \in \mu^{-1}(t)$ for some $t \in (0, \mu(X)]$. Then there exists $y \in B \setminus A$. For every $x \in A$ there is a continuum A_x such that $x \in \text{int } A_x$ and $y \notin A_x$. Now $\{\text{int } A_x | x \in A\}$ is an open cover of A and A is compact, so there is a finite subcover $\{\text{int } A_{x_i} | i = 1, \cdots, n\}$. Let $A' = \bigcup_{i=1}^n A_{x_i}$. Then $A \subseteq \bigcup_{i=1}^n (\text{int } A_{x_i}) \subseteq \text{int } \bigcup_{i=1}^n A_{x_i} = \text{int } A'$, and so A is an element of the interior of the continuum $C(A',t)$. Since $y \in B \setminus A_x$ for all $x \in A$, we have $B \not\subseteq A'$ and hence $B \in \mu^{-1}(t) \setminus C(A',t)$. Therefore $\mu^{-1}(t)$ is aposyndetic at A with respect to B, and it follows that $\mu^{-1}(t)$ is aposyndetic.

Proposition 10. Finite aposyndesis is a Whitney property.

Proof. Suppose X is finitely aposyndetic, $\{A_1,\cdots,A_n,A\} \subseteq \mu^{-1}(t)$ for some $t \in (0,\mu(X))$, and $A \neq A_i$ for all $i = 1,\cdots,n$. For each $i = 1,\cdots,n$, let $a_i \in A_i \backslash A$. Then for each $a \in A$ there exists a continuum M_a such that $a \in \text{int } M_a$ and $M_a \cap \{a_1,\cdots,a_n\} = \emptyset$. The remainder of the proof is exactly like that of the preceeding proposition.

Proposition 11. Mutual aposyndesis is a Whitney property.

Proof. Suppose X is mutually aposyndetic. Let $A_1, A_2 \in \mu^{-1}(t)$ for some $t \in (0,\mu(X))$. Let $a_1 \in A_1 \backslash A_2$ and $a_2 \in A_2 \backslash A_1$. Then there exist disjoint continua A_1' and A_2' such that for some η, $B_d(a_i;\eta) \subseteq A_i'$ for $i = 1, 2$. Let $\delta < \min\{\eta,\ d(a_1,A_2'),\ d(a_2,\ A_1'),\ \frac{1}{3}d(a_2,A_1)\}$. We will construct disjoint continua $\mathfrak{A}_1$ and $\mathfrak{A}_2$ in $\mu^{-1}(t)$ such that $B_\rho(A_i;\delta) \cap \mu^{-1}(t) \subseteq \mathfrak{A}_i$, for $i = 1, 2$.

Let $i = 1$ or 2. We consider 2 cases:

Case I: Suppose that $\mu(A_i') < t$. Let $\mathfrak{B}_i = \{A_i''\}$, where A_i'' is some element of $\mu^{-1}(t)$ such that $A_i' \subseteq A_i'' \subseteq A_i' \cup A_i$. For each $D_i \in \mu^{-1}(t) \cap B_\rho(A_i,\delta)$, there exists $g_i(D_i) \in D_i \cap B_d(a_i;\delta) \subseteq D_i \cap A_i''$. Then by Lemma 0 there is an arc $\{D_i^s\}$ in $\mu^{-1}(t)$ from D_i to A_i'' such that $g_i(D_i) \in D_i^s$ and $D_i^s \subseteq D_i \cup A_i''$ for all s. Let $\mathfrak{D}_i = \cup\{\{D_i^s\} | D_i \in \mu^{-1}(t) \cap B_\rho(A_i;\delta)\}$. Let $\mathfrak{A}_i = \mathfrak{B}_i \cup \overline{\mathfrak{D}}_i$. Then $\mu^{-1}(t) \cap B_\rho(A_i;\delta) \subseteq \mathfrak{A}_i$. To show that $\mathfrak{A}_i$ is a continuum, note that each point of $\mathfrak{D}_i$ is contained in an arc which contains the point $A_i'' = \mathfrak{B}_i$, so $\mathfrak{B}_i \cup \mathfrak{D}_i$ is connected. Hence $\mathfrak{A}_i = \mathfrak{B}_i \cup \overline{\mathfrak{D}}_i$ is a continuum.

Case II: Suppose that $\mu(A_i') \geq t$. Let $\mathfrak{B}_i = C(A_i',t)$. We construct arcs as in Case I except that each arc is from D_i to some point $D_i' \in C(A_i',t)$ such that $g_i(D_i) \in D_i^s \subseteq D_i \cup D_i'$ for all s. Let $\mathfrak{D}_i$ be defined as above and let $\mathfrak{A}_i = \mathfrak{B}_i \cup \overline{\mathfrak{D}}_i$. Again we have $\mu^{-1}(t) \cap B_\rho(A_i;\delta) \subseteq \mathfrak{A}_i$. Note that $\mathfrak{B}_i$ is a continuum and each point of $\mathfrak{D}_i$ is contained in an arc which intersects $\mathcal{B}_i$. So $\mathcal{B}_i \cup \mathfrak{D}_i$ is connected. Hence $\mathfrak{A}_i = \mathcal{B}_i \cup \overline{\mathfrak{D}}_i$ is a continuum.

We now show that $\mathfrak{A}_1 \cap \mathfrak{A}_2 = \emptyset$. Note that any element of $\mathfrak{A}_i$, i = 1, 2, either is contained in A_i' or intersects $\overline{B_d(a_i;\delta)}$. Suppose $F \in \mathfrak{A}_1$. If $F \in \mathcal{B}_1$, then either

(1) $F \subseteq A_1'$, in which case $F \cap \overline{B_d(a_2;\delta)} = \emptyset$ since $\delta < d(a_2,A_1')$, and $F \not\subseteq A_2'$ since $A_1' \cap A_2' = \emptyset$, so $F \notin \mathfrak{A}_2$, or

(2) $F = A_1''$. Then $A_1' \subseteq F \subseteq A_1' \cup A_1$. So $F \not\subseteq A_2'$ since $A_1' \cap A_2' = \emptyset$. Also $F \cap \overline{B_d(a_2;\delta)} = \emptyset$ since $d(a_2,A_1) > \delta$ and $d(a_2,A_1') > \delta$. Thus $F \notin \mathfrak{A}_2$.

So assume $F \in \overline{\mathfrak{D}}_1$. Then $F \subseteq A_1' \cup \overline{V_\delta(A_1)}$ and $F \cap \overline{B(a_1;\delta)} \neq \emptyset$. Then $F \not\subseteq A_2'$ since $d(a_1,A_2') > \delta$. Now $A_1' \cap \overline{B_d(a_2;\delta)} = \emptyset$ since $d(a_2,A_1') > \delta$, and $\overline{V_\delta(A_1)} \cap \overline{B(a_2;\delta)} = \emptyset$ since $\delta < \frac{1}{3}d(a_2,A_1)$. Thus $F \cap \overline{B(a_2;\delta)} = \emptyset$. Hence $F \notin \mathfrak{A}_2$. Therefore $\mathfrak{A}_1 \cap \mathfrak{A}_2 = \emptyset$. So we have $\mu^{-1}(t)$ mutually aposyndetic at the pair (A_1,A_2).

The method used in constructing the continua $\mathfrak{A}_i$ in the preceeding proof can be used in proving the following proposition.

Proposition 12. Semi-aposyndesis is a Whitney property.

Proof. Suppose X is semi-aposyndetic. Let $A,B \in \mu^{-1}(t)$ for some $t \in (0,\mu(X))$. Let $a \in A\backslash B$ and $b \in B\backslash A$. There exists a continuum M such that for one of a and b, say a, $a \in \text{int } M$ and $b \notin M$. Let $M = A'$. There exists η such that $B_d(a;\eta) \subseteq A'$. Let $\delta_o < \min\{\eta, \frac{1}{3}d(a,B), \frac{1}{3}d(b,A), d(b,A')\}$. Now by the method of Proposition 11, using A for A_i, A' for A_i', and δ_o for δ, we can construct a continuum $\mathfrak{A}$ in $\mu^{-1}(t)$ such that $A \in \text{int } \mathfrak{A}$ and $B \notin \mathfrak{A}$. Thus $\mu^{-1}(t)$ is semi-aposyndetic at the pair (A,B).

3. A Covering Property of Whitney Continua

In [8] Krasinkiewicz and Nadler raise the following question: What continua X have the property that if $\mathcal{A}$ is a subcontinuum of $\mu^{-1}(t)$, $t \in [0,\mu(X)]$, and $\sigma\mathcal{A} = X$, then $\mathcal{A} = \mu^{-1}(t)$? It is shown in [8] that arc-like and hereditarily indecomposable continua have this property, which will be referred to as (1). We will also consider a stronger property, namely (2) if $\mathcal{A}$ is a subcontinuum of $\mu^{-1}(t)$, $t \in [0,\mu(X)]$, then $\mathcal{A} = C(\sigma\mathcal{A},t) = C(\sigma\mathcal{A}) \cap \mu^{-1}(t)$. What (2) says is that each subcontinuum A of X has (1) for $t \in [0,\mu(A)]$. We note first that (1) and (2) are actually properties of the space X and do not depend on the choice of the Whitney map μ.

Proposition 13. Let μ_1 *and* μ_2 *be Whitney maps on* C(X). *If* X *has* (1) *with respect to* μ_1, *then* X *has* (1) *with respect to* μ_2.

Proof. Let $s \in [0,\mu_2(X)]$ and $\mathcal{A} \in C(\mu_2^{-1}(s))$ such that $\mathcal{A} \neq \mu_2^{-1}(s)$. Choose $A_0 \in \mu_2^{-1}(s)\setminus\mathcal{A}$. Suppose $\mu_1(A_0) = t$. Let $\mathcal{B} = \{B \in \mu_1^{-1}(t) \mid \text{for some } A \in \mathcal{A},\ A \subseteq B \text{ or } B \subseteq A\}$. If $A \in \mathcal{A}$ and $\mu_1(A) \geq t$, then for each $a \in A$ there is a $B \in \mu_1^{-1}(t)$ such that $a \in B \subseteq A$. If $A \in \mathcal{A}$, $\mu_1(A) < t$, then there is a $B \in \mu_1^{-1}(t)$ such that $A \subseteq B$. Hence $\sigma\mathcal{A} \subseteq \sigma\mathcal{B}$. Now $\mathcal{B} \neq \mu_1^{-1}(t)$ since $A_0 \in \mu_1^{-1}(t)\setminus\mathcal{B}$. So if $\mathcal{B}$ is a continuum, then by (1) for μ_1 we will have that $\sigma\mathcal{B} \neq X$ and $\sigma\mathcal{A} \neq X$. This shows that X has (1) for μ_2. Hence it remains only to show that $\mathcal{B}$ is a continuum.

Let $\{B_n\}$ be a sequence of elements of $\mathcal{B}$ such that $B_n \to B$. With each B_n there is associated an $A_n \in \mathcal{A}$ such that $A_n \subseteq B_n$ or $B_n \subseteq A_n$. Some subsequence $\{A_{n_i}\}$ of $\{A_n\}$ converges to $A \in \mathcal{A}$. Then, since $B_{n_i} \to B$, we have $A \subseteq B$ or $B \subseteq A$, and so $B \in \mathcal{B}$. Thus $\mathcal{B}$ is closed.

Finally, we show that $\mathcal{B}$ is connected. Suppose not. Then there exist disjoint, non-empty sets $\mathcal{B}_1$ and $\mathcal{B}_2$, open in $\mathcal{B}$, such that $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$. Let $\mathcal{A}_i = \{A \in \mathcal{A} \mid \text{for some } B \in \mathcal{B}_i,$ $A \subseteq B \text{ or } B \subseteq A\}$, $i = 1, 2$. Suppose $A \in \mathcal{A}_1 \cap \mathcal{A}_2$. If $\mu_1(A) < t$, there exist $B_1 \in \mathcal{B}_1$, $B_2 \in \mathcal{B}_2$ such that $A \subseteq B_1 \cap B_2$. By Lemma 0 there is an arc $\{B_j\}$ in $\mu_1^{-1}(t)$ from B_1 to B_2 such that $A \subseteq B_j$ for all j. But then $\{B_j\}$ is an arc in $\mathcal{B}$ from a point of $\mathcal{B}_1$ to a point of $\mathcal{B}_2$. This is impossible.

So assume $\mu_1(A) \geq t$. Then there exist $B_1 \in \mathcal{B}_1$, $B_2 \in \mathcal{B}_2$ with $B_1, B_2 \subseteq A$, so that $C(A) \cap \mu_1^{-1}(t)$ is a continuum contained in $\mathcal{B}$ and intersecting both $\mathcal{B}_1$ and $\mathcal{B}_2$. This is also a contradiction, so we must have $\mathcal{A}_1 \cap \mathcal{A}_2 = \emptyset$. Since $\mathcal{B}_1$ and $\mathcal{B}_2$ are non-empty, so are $\mathcal{A}_1$ and $\mathcal{A}_2$. Since $\mathcal{B}_1 \cup \mathcal{B}_2 = \mathcal{B}$, we have $\mathcal{A}_1 \cup \mathcal{A}_2 = \mathcal{A}$.

To reach the contradiction that $\mathcal{A}$ is not connected, it remains to show that $\mathcal{A}_1$ and $\mathcal{A}_2$ are open in $\mathcal{A}$. Let $\{A_n\}$ be a sequence of elements of $\mathcal{A}_2$ such that $A_n \to A$. With each A_n there is associated a $B_n \in \mathcal{B}_2$ such that $A_n \subseteq B_n$ or $B_n \subseteq A_n$. Since $\mathcal{B}_2$ is closed in $\mathcal{B}$, some subsequence of $\{B_n\}$ converges to an element $B \in \mathcal{B}_2$ and we must have $A \subseteq B$ or $B \subseteq A$. Thus $A \in \mathcal{A}_2$ and we conclude that $\mathcal{A}_2$ is closed in $\mathcal{A}$. Hence $\mathcal{A}_1$ is open in $\mathcal{A}$. By a symmetric argument $\mathcal{A}_2$ is open in $\mathcal{A}$. But $\mathcal{A}$ is a continuum, so $\mathcal{B}$ must be connected and hence a continuum.

Corollary 14. If X *has* (2) *with respect to some Whitney map* μ, *then* X *has* (2) *with respect to any Whitney map.*

Proof. For $A \in C(X)$, $\mu|C(A)$ is a Whitney map. Apply the proposition to each $A \in C(X)$.

Note that (1) and (2) are not equivalent since a ray spiraling down on a circle is a continuum which has (1) but not (2).

We may view (1) and (2) in terms of the maps σ_t which have been considered in [11]. For a given continuum X and Whitney map μ on $C(X)$, if $t \in [0,\mu(X)]$, then $\sigma_t\colon C(\mu^{-1}(t)) \to \mu^{-1}([t,\mu(X)])$

is defined to be $\sigma|C(\mu^{-1}(t))$. We define $\phi_t\colon \mu^{-1}([t,\mu(X)]) \to C(\mu^{-1}(t))$ by $\phi_t(A) = C(A,t)$ for $A \in \mu^{-1}([t,\mu(X)])$. Then (1) is equivalent to $\sigma_t^{-1}(X) = \mu^{-1}(t)$ for all $t \in [0,\mu(X)]$, and (2) is equivalent to the injectivity of σ_t for all $t \in [0,\mu(X)]$.

Proposition 15. If σ_t *is injective for some* $t \in [0,\mu(X)]$, *then* σ_t *is a homeomorphism and* $\sigma_t^{-1} = \phi_t$.

Proof. The function σ_t is continuous because it is a restriction of the map σ. If $A \in \mu^{-1}([t,\mu(X)])$, then $\sigma_t(C(A,t)) = A$, so σ_t is surjective. But $C(\mu^{-1}(t))$ is compact and $\mu^{-1}([t,\mu(X)])$ is Hausdorff, so σ_t is a homeomorphism.

Corollary 16. If X *has* (2), *then* $\mu^{-1}([t,\mu(X)] \cong C(\mu^{-1}(t))$ *for all* $t \in [0,\mu(X)]$.

If σ_t is injective and hence a homeomorphism, then $\phi_t = \sigma_t^{-1}$ and so ϕ_t is continuous. However, if σ_t is not injective, then ϕ_t need not be continuous, as in the case $X = S^1$ for any $t \in (0,\mu(X))$.

The following proposition shows that even when σ_t is not injective, point inverses are nice continua.

Proposition 17. For any $A \in \mu^{-1}([t,\mu(X)])$, $\sigma_t^{-1}(A)$ *is an arc-connected, acyclic continuum.*

Proof. The set $\sigma_t^{-1}(A)$ is closed, since σ_t is continuous. If $\mathcal{A} \in \sigma_t^{-1}(A)$, then $\mathcal{A} \subseteq C(A,t)$, so there is a segment in $C(\mu^{-1}(t))$ from $\mathcal{A}$ to $C(A,t)$. If $\mathcal{B}$ is any point of the segment, then $\mathcal{A} \subseteq \mathcal{B} \subseteq C(A,t)$, so $A = \sigma_t(\mathcal{A}) \subseteq \sigma_t(\mathcal{B}) \subseteq \sigma_t(C(A,t)) = A$ and thus $\sigma_t(\mathcal{B}) = A$. Hence the segment is an arc in $\sigma_t^{-1}(A)$ from $\mathcal{A}$ to $C(A,t)$. So $\sigma_t^{-1}(A)$ is an arc-connected continuum.

By Theorem 2 of [13], for any closed set $\mathcal{B} \in C(C(A,t))$, $M(\mathcal{B}) = \{\mathcal{Y} \in C(C(A,t)) \mid \mathcal{Y}$ contains a subcontinuum of $C(A,t)$ which is a point of $\mathcal{B}\}$ is acyclic. Let $\mathcal{A} = \{\mathcal{Y} \in C(C(A,t)) \mid \sigma_t \mathcal{Y} = A$ and if $\mathcal{Y}' \subsetneq \mathcal{Y}$, then $\sigma_t \mathcal{Y}' \subsetneq A\}$.

Then $M(\overline{\mathfrak{A}})$ is acyclic. We show that $M(\overline{\mathfrak{A}}) = \sigma_t^{-1}(A)$. Let $\mathfrak{E} \in M(\overline{\mathfrak{A}})$. Then $\mathfrak{E}$ contains some $\mathcal{Y}_0 \in \overline{\mathfrak{A}}$. But $\sigma_t(\mathcal{Y}) = A$ for all $\mathcal{Y} \in \mathfrak{A}$, so by the continuity of σ_t, $\sigma_t(\mathcal{Y}_0) = A$. Thus $A = \sigma_t(\mathcal{Y}_0) \subseteq \sigma_t(\mathfrak{E}) \subseteq \sigma_t(C(A,t)) = A$ and so $\mathfrak{E} \in \sigma_t^{-1}(A)$. Hence $M(\mathfrak{A}) \subseteq \sigma_t^{-1}(A)$. If $\mathcal{Y} \in \sigma_t^{-1}(A)$, then any segment from a point of $\mathcal{Y}$ to $\mathcal{Y}$ will contain a point of $\mathfrak{A}$. So $\mathcal{Y} \in M(\mathfrak{A}) \subseteq M(\overline{\mathfrak{A}})$. Hence $M(\overline{\mathfrak{A}}) = \sigma_t^{-1}(A)$ and $\sigma_t^{-1}(A)$ is acyclic.

We turn now to the question of which continua have properties (1) and (2). The next proposition follows immediately from a result of Krasinkiewicz and Nadler [8].

Proposition 18. Arc-like continua and hereditarily indecomposable continua have property (2).

Proof. It is shown in [8] that these continua satisfy (1). Subcontinua of an arc-like (hereditarily indecomposable) continuum are also arc-like (hereditarily indecomposable). Therefore arc-like continua and hereditarily indecomposable continua satisfy (2).

Proposition 19. If X *is a non-planar circle-like continuum, then* X *has property* (2).

Proof. If A is a proper subcontinuum of X, then A is arc-like and hence has (1). To show that X has (1), let $\mathfrak{A}$ be a proper subcontinuum of $\mu^{-1}(t)$, $0 \leq t < \mu(X)$. Now $\mu^{-1}(t)$ is circle-like [14], so $\mathfrak{A}$ is arc-like. Suppose $\sigma_t \mathfrak{A} = X$. The function σ_t can be considered as a continuum-valued function from $\mu^{-1}(t)$ to X, and it has been shown by Rogers in [14] to be upper semi-continuous. Then since $\sigma_t(\mathfrak{A}) = X$, Theorem 7 of [1] implies that for some $A \in \mathfrak{A}$, $\sigma_t(A)$ is not a proper subcontinuum of X. But $\sigma_t(A) = A$ and A is a proper subcontinuum of X for all $A \in \mathfrak{A}$. Therefore $\sigma_t \mathfrak{A} \neq X$ and so X satisfies (1). Hence X satisfies (2).

Although the case of X being a solenoid is covered by the preceding proposition, we include a particularly simple proof that solenoids satisfy (2).

Proposition 20. If X *is a solenoid, then* X *has property* (2).

Proof. Subcontinua of X are arcs and so have (1). Let $\mathfrak{A} \in C(\mu^{-1}(t))$, $\mathfrak{A} \neq \mu^{-1}(t)$, $0 \leq t < \mu(X)$. Then, since $\mu^{-1}(t)$ is a solenoid ([8]), $\mathfrak{A}$ is an arc. Suppose $\sigma_t \mathfrak{A} = X$. Then by Lemma 8.1 of [6], $X \in \mathfrak{A}$ since X is indecomposable. But $X \notin \mathfrak{A}$, so X has property (1). Hence X satisfies (2).

In [8] it is mentioned that (1) implies the unicoherence of X. We provide proofs that (2) implies that X is both hereditarily unicoherent and atriodic.

Proposition 21. If S *satisfies* (2), *then* X *is hereditarily unicoherent.*

Proof. Let Y be a subcontinuum of X and suppose $Y = A \cup B$, $A,B \in C(X)$, $A \cap B$ not connected. Let M and N be components of $A \cap B$, and let γ_M and γ_N be segments from M and N to A. Choose t_1 and t_2 so that $\mu(M), \mu(N) < t_1 < t_2 < \mu(A), \mu(B)$ and $(\gamma_M \cap \mu^{-1}(t_1)) \cap (\gamma_N \cap \mu^{-1}(t_2)) = \emptyset$. By using a segment from M to B it is possible to find $M' \in \mu^{-1}(t_2)$ so that $M' \cap A = \gamma_M \cap \mu^{-1}(t_1)$. Then $M' \cap (B \setminus A) \neq \emptyset \neq M' \cap (A \setminus B)$, so $M' \notin C(A,t_2)$, $M' \notin C(B,t_2)$. Let $N' = \gamma_N \cap \mu^{-1}(t_2)$. By Lemma 0 there is an arc $\{D_j\}$ in $\mu^{-1}(t_2)$ from $D_0 = N'$ to $D_1 \subseteq B$ such that $D_j \cap A \subseteq N'$ for all $j \in [0,1]$. Then $M' \notin \{D_j\}$ since $M' \cap A$ is disjoint from N'. So $C(A,t_2) \cup C(B,t_2) \cup \{D_j\}$ is a continuum in $C(Y,t_2)$ not containing the point $M' \in C(Y,t_2)$. But $\sigma_{t_2}(C(A,t_2) \cup C(B,t_2) \cup \{D_j\}) = Y$ and this contradicts the fact that Y has property (1). Therefore Y is unicoherent and X is hereditarily unicoherent.

Proposition 22. If X *has property* (2), *then* X *is atriodic.*

Proof. Suppose X contains a triod $T = A_1 \cup A_2 \cup A_3$, $A_i \in C(X)$, $A_i \cap A_j = Y$, $A_i \setminus Y \neq \emptyset$ for all $i,j \in \{1,2,3\}$, $i \neq j$. Without loss of generality assume that $\mu(A_i) = t_0$ for all i. Choose t so that $\mu(Y) < t < t_0$. Let $\mathcal{A} = C(A_1 \cup A_3,t) \cup C(A_2 \cup A_3,t)$. Then $\sigma\mathcal{A} = T$ and $\mathcal{A}$ is a continuum since $C(A_3,t) \subseteq C(A_1 \cup A_3,t) \cap C(A_2 \cup A_3,t)$. But $\mathcal{A} \neq C(T,t)$ since there are points of $C(A_1 \cup A_2,t)$ which are not contained in $\mathcal{A}$. This contradicts the fact that X satisfies (2). Hence X is atriodic.

Example 23. Let X be an hereditarily indecomposable arc-like continuum with a pair of opposite endpoints identified. Then X is indecomposable and circle-like and also hereditarily unicoherent and atriodic. Let p be the point of X which is obtained by the identification of endpoints. It has been shown, [8, 14], that for any t with $0 < t < \mu(X)$, the set $C_p^t = \{A \in \mu^{-1}(t) \mid p \in A\}$ is an arc in $\mu^{-1}(t)$ with non-empty interior. The set $\overline{\mu^{-1}(t) \setminus C_p^t}$ is a proper subcontinuum of $\mu^{-1}(t)$ and $\sigma_t(\overline{\mu^{-1}(t) \setminus C_p^t}) = X$. So X does not have property (1).

Example 24. Planar, circle-like, non-arc-like continua may satisfy (2). Take X to be the pseudo-circle.

Continua which have properties (1) and (2) have some special characteristics with regard to Whitney properties.

Lemma 25. If for some $t \in [0,\mu(X)]$, *the map* σ_t *is injective, then there is a Whitney map* μ_0 *on* $C(\mu^{-1}(t))$ *so that* ϕ_t *preserves Whitney continua, i.e. for any* $s_1 \in [t,\mu(X)]$, $\phi_t(\mu^{-1}(s_1)) = \mu_0^{-1}(s_2)$ *for some* $s_2 \in [0,\mu_0(C(\mu^{-1}(t)))]$.

Proof. For $\mathcal{A} \in C(\mu^{-1}(t))$, let $\mu_0(\mathcal{A}) = \mu(\sigma\mathcal{A}) - t$. Then μ_0 is a continuous function from $C(\mu^{-1}(t))$ into $[0,\infty)$ such that $\mu_0(A) = 0$ for all $A \in \mu^{-1}(t)$. If $\mathcal{A}, \mathcal{B} \in C(\mu^{-1}(t))$ such that $\mathcal{A} \subsetneq \mathcal{B}$, then since σ_t is injective, $\sigma\mathcal{A} \subsetneq \sigma\mathcal{B}$ and so

$\mu(\sigma\mathfrak{A}) < \mu(\sigma\mathfrak{B})$. Thus $\mu_0(\mathfrak{A}) < \mu_0(\mathfrak{B})$, and it follows that μ_0 is a Whitney map.

If $A,B \in \mu^{-1}([t,\mu(X)])$ and $\mu(A) = \mu(B)$, then $\sigma(\phi_t(A)) = A$ and $\sigma(\phi_t(B)) = B$, so $\mu_0(\phi_t(A)) = \mu(A) - t = \mu(B) - t = \mu_0(\phi_t(B))$. If $\mathfrak{A}, \mathfrak{B} \in C(\mu^{-1}(t))$ and $\mu_0(\mathfrak{A}) = \mu_0(\mathfrak{B})$, then $\mu(\sigma\mathfrak{A}) = \mu(\sigma\mathfrak{B})$. But $\sigma\mathfrak{A} = \phi_t^{-1}(\mathfrak{A})$ and $\sigma\mathfrak{B} = \phi_t^{-1}(\mathfrak{B})$ since σ_t is injective, so $\mu(\phi_t^{-1}(\mathfrak{A})) = \mu(\phi_t^{-1}(\mathfrak{B})) = (\phi_t^{-1}(\mathfrak{B}))$. Hence ϕ_t preserves Whitney continua.

Corollary 26. Let $t_0 \in [0,\mu(X)]$, σ_{t_0} *be injective,* P *be a Whitney property such that* $\mu^{-1}(t_0)$ *has* P, *and* $t > t_0$. *Then* $\mu^{-1}(t)$ *has* P.

Corollary 27. If X *satisfies* (2) *and* $\mu^{-1}(t_0)$ *has a Whitney property* P, *then* $\mu^{-1}(t)$ *has* P *for all* $t \in [t_0,\mu(X)]$.

Although indecomposability is not a Whitney property, we have the following:

Proposition 28. If X *is indecomposable and* $\sigma_t^{-1}(X) = \{\mu^{-1}(t)\}$ *for some* $t \in (0,\mu(X))$, *then* $\mu^{-1}(t)$ *is indecomposable. If* X *is indecomposable and has* (1), *then* $\mu^{-1}(t)$ *is indecomposable for all* $t \in [0,\mu(X)]$.

Proof. Suppose $\sigma_t^{-1}(X) = \{\mu^{-1}(t)\}$ and $\mu^{-1}(t) = \mathfrak{A} \cup \mathfrak{B}$ is a decomposition. Then $\sigma_t(\mathfrak{A}) \neq X \neq \sigma_t(\mathfrak{B})$ and $X = \sigma_t(\mathfrak{A}) \cup \sigma_t(\mathfrak{B})$ is a decomposition of X. So $\mu^{-1}(t)$ is indecomposable. If X satisfies (1), then $\sigma_t^{-1}(X) = \{\mu^{-1}(t)\}$ for all $t \in [0,\mu(X)]$.

We note that the proof of the preceding proposition is the same as that used in [8] to show that being an indecomposable chainable continuum is a Whitney property.

4. Essentially Different Whitney Maps

In [10] Nadler raised the question of whether the topological types of Whitney map inverses is a topological invariant. In other words, if X and Y are homeomorphic and μ_1 and μ_2 are Whitney maps on C(X) and C(Y) respectively, then for each $s \in [0,\mu_2(Y)]$, does there exist a $t \in [0,\mu_1(X)]$ such that $\mu_1^{-1}(t)$ is homeomorphic to $\mu_2^{-1}(s)$? We will answer this question negatively, give some examples, and make some statements about the types of spaces for which the answer may be negative.

Definition. Let X be a continuum. If there exist a continuum Y homeomorphic to X, Whitney maps μ_1 and μ_2 on C(X) and C(Y) respectively, and $s \in [0,\mu_2(Y)]$ such that for every $t \in [0,\mu_1(X)]$, $\mu_2^{-1}(s)$ is not homeomorphic to $\mu_1^{-1}(t)$, then we will say that X admits essentially different Whitney maps.

Examples 29. (a) Let $X = \bigcup_{i=1}^{3} A_i$, where A_i is an arc for each i, and $A_i \cap A_j = \{p\}$, $p \in X$, for $i \neq j$. Suppose μ_1 is a Whitney map on C(X) such that $\mu_1(A_1) = \mu_1(A_2) = \mu_1(A_3)$. Then for each $t \in [0,\mu_1(X)]$, $\mu_1^{-1}(t)$ is homeomorphic to one of the following: X, the union of a 2-cell and three mutually disjoint arcs each intersecting the 2-cell in a single point, a 2-cell, a point. Define μ_2: $C(X) \to [0,\infty)$ by $\mu_2(A) = \mu_1(A \cap A_1) + \mu_1(A \cap A_2) + 2\,\mu_1(A \cap A_3)$. Then μ_2 is a Whitney map and there exists $s \in (0,\mu_2(X))$ such that $\mu_2^{-1}(s)$ is the union of an arc and a 2-cell which intersect in exactly one point. Thus $\mu_2^{-1}(s)$ fails to be homeomorphic to $\mu_1^{-1}(t)$ for all $t \in [0,\mu_1(X)]$, and so X admits essentially different Whitney maps.

(b) Using the same method as in (a) we can show that X admits essentially different Whitney maps if X is the union of three hereditarily indecomposable continua A_i, i = 1, 2, 3, joined together at a point p. Let μ_1 and μ_2 be as in (a). Then for $t \in [0,\mu_1(X)]$, $\mu_1^{-1}(t)$ is homeomorphic to one of the following: X, the union of a 2-cell and three mutually disjoint hereditarily indecomposable continua each attached to the 2-cell at a point, a 2-cell, a point. But for some $s \in (0,\mu_2(X))$, $\mu_2^{-1}(s)$ is the union of a 2-cell and an hereditarily indecomposable continuum with one-point intersection.

(c) For X an arc, a circle, a pseudo-arc, or a pseudo-circle, each Whitney continuum is C(X) is homeomorphic to X [7, 2, 14]. So X does not admit essentially different Whitney maps.

(d) Let X be a ray spiralling down on a circle or the $\sin \frac{1}{x}$ continuum $\{(x,y) | x \in (0,1], y = \sin \frac{1}{x}\} \cup (\{0\} \times [-1,1])$. In either case the topological types of the Whitney continua are exactly those of X, an arc, and a point for any Whitney map. So X does not admit essentially different Whitney maps.

(e) Let X be the continuum pictured in Figure 1. Given analytically, $X = X_1 \cup X_2$ where $X_1 = \{(x,y) | x \in (0,1], y = \sin \frac{1}{x}\} \cup (\{0\} \times [-1,1])$ and $X_2 = \{(x,y) | x \in [-1,0), y = 2 + \sin \frac{1}{x}\} \cup (\{0\} \times [1,3])$. Let μ_1 be a Whitney map on C(X) such that $\mu_1(\{0\} \times [-1,1]) = \mu_1(\{0\} \times [1,3])$. Rogers has noted [14] that for $t \in [0,\mu_1(X)]$, $\mu_1^{-1}(t)$ is homeomorphic to X, an arc, a point, or the space shown in Figure 2. Now define μ_2: $C(X) \to [0,\infty]$ by $\mu_2(A) = 2\,\mu_1(A \cap X_1) + \mu_1(A \cap X_2)$. Then μ_2 is a Whitney map and for some $s \in (0,\mu_2(X))$, $\mu_2^{-1}(s)$ is homeomorphic to the space shown in Figure 3. So X does admit essentially different Whitney maps.

(f) Let X' be the continuum X of example (e) with the points $(-1, 2 + \sin(-1))$ and $(1, \sin 1)$ joined by an arc which is otherwise disjoint from X. Then X' also admits essentially different Whitney maps.

We see from the above examples that the collection of continua which admit essentially different Whitney maps includes continua which are triods (a), atriodic (e), 3-indecomposable

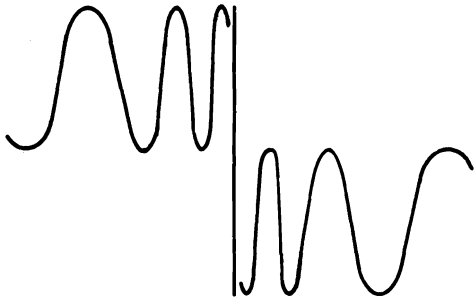

Figure 1

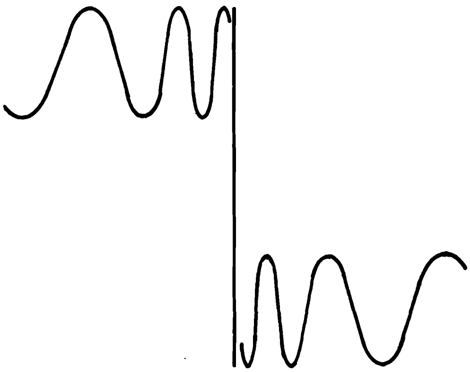

Figure 2

Figure 3

(b), arc-like (e), circle-like (f), arc-connected (a), non arc-connected (e), unicoherent (e), and non unicoherent (f). Among the continua which do not admit essentially different Whitney maps are included ones which are decomposable (d), hereditarily indecomposable (c), arc-like (d), circle-like (c), arc-connected (c), non arc-connected (d), unicoherent (c), and non unicoherent (c).

We now make some generalizations about continua which admit essentially different Whitney maps.

Proposition 30. Let Y *be a decomposable, non arc-connected continuum with finitely many arc components, and let* $p \in Y$. *Let* X_1 *and* X_2 *be copies of* Y *and let* $X = X_1 \cup X_2/\{p\}$. *Then* X *admits essentially different Whitney maps.*

Proof. Let μ be a Whitney map on $C(Y)$. Since Y is decomposable, by Corollary 4 there exists $t_o \in (0,\mu(X))$ such that $\mu^{-1}(t)$ is arc-connected if $t > t_o$ and non arc-connected if $t < t_o$. For i = 1, 2, designate μ on $C(X_i)$ by μ_i. Define a Whitney map $\bar{\mu}$ on $C(X)$ by $\bar{\mu}(K) = \mu_1(K \cap X_1) + \mu_2(K \cap X_2)$ for $K \in C(X)$. For $t \in [0,\bar{\mu}(X)]$, let $K_t = \{K \in \bar{\mu}^{-1}(t) | p \in K\}$. Then $\bar{\mu}^{-1}(t) = \mu_1^{-1}(t) \cup \mu_2^{-1}(t) \cup K_t$.

For any continuum M with finitely many arc components, let n(M) be the number of arc components of M.

We now examine $n(\bar{\mu}^{-1}(t))$ for $t \in [0,\bar{\mu}(X)]$. For t = 0, $\bar{\mu}^{-1}(t) = X$ and $n(\bar{\mu}^{-1}(t)) = n(X_1) + n(X_2) - 1 = 2\ n(Y) - 1 = 2\ n(\mu^{-1}(t)) - 1$. For $t > 0$, we have that K_t is arc-connected [8] and $\mu_i^{-1}(t) \cap K_t = \{K \in \mu_i^{-1}(t) | p \in K\}$ is arc-connected for i = 1,2 [8]. Also $\mu_1^{-1}(t) \cap \mu_2^{-1}(t) = \emptyset$. So since $\bar{\mu}^{-1}(t) = \mu_1^{-1}(t) \cup \mu_2^{-1}(t) \cup K_t$, we have $n(\bar{\mu}^{-1}(t)) = n(\mu_1^{-1}(t)) + n(\mu_2^{-1}(t)) - 1 = 2\ n(\mu^{-1}(t)) - 1$.

By Corollary 3, the function n restricted to the set $\{\mu^{-1}(t) | t \in [0,\mu(Y)]\}$ is a decreasing, positive integer-valued function of t. The continuum $\mu^{-1}(t_o)$ may or may not be arc-connected. If $n(\mu^{-1}(t_o)) \neq 1$, let $n_o = n(\mu^{-1}(t_o))$. Otherwise, let $n_o = \min\{n(\mu^{-1}(t)) | t \in [0,t_o)\}$. Note that $n_o \geq 2$. Now for $0 \leq t < t_o$, $n(\bar{\mu}^{-1}(t)) \geq 2n_o - 1$; for $t = t_o$, $n(\bar{\mu}^{-1}(t)) = 1$ or $2n_o - 1$; for $t > t_o$, $\mu^{-1}(t)$ is arc-connected, so $n(\bar{\mu}^{-1}(t)) = 2(1) - 1 = 1$.

Define a Whitney map $\tilde{\mu}$ on C(X) by $\tilde{\mu}(A) = \mu_1(A \cap X_1) + 2\mu_2(A \cap X_2)$. Then $\tilde{\mu}^{-1}(t) = \mu_1^{-1}(t) \cup \mu_2^{-1}(\frac{1}{2}t) \cup A_t$ where $A_t = \{A \in \tilde{\mu}^{-1}(t) | p \in A\}$. If $0 \leq t \leq \mu_1(X)$, then as in the case of $\bar{\mu}$ above, we get $n(\tilde{\mu}^{-1}(t)) = n(\mu_1^{-1}(t)) + n(\mu_2^{-1}(\frac{1}{2}t)) - 1$. If $t > \mu_1(X)$, then $\mu_1^{-1}(t) = \emptyset$, so $\tilde{\mu}^{-1}(t) = \mu_2^{-1}(\frac{1}{2}t) \cup A_t$. But A_t is arc-connected and $\mu_2^{-1}(\frac{1}{2}t) \cap A_t$ is arc-connected (as argued previously for K_t and $\mu_2^{-1}(t) \cap K_t$), so for $t > \mu_1(X_1)$, we have $n(\tilde{\mu}^{-1}(t)) = n(\mu_2^{-1}(\frac{1}{2}t))$.

Suppose $s \in (t_o, 2t_o]$. Then $\mu_1^{-1}(s)$ is either empty or an arc-connected continuum. Since $\frac{1}{2}t_o < \frac{1}{2}s \leq t_o$, s can be chosen so that $n(\mu_2^{-1}(\frac{1}{2}s)) = n(\mu^{-1}(\frac{1}{2}s)) = n_o$. Let s_o be such a value of s. If $s_o \leq \mu_1(X_1)$, then $n(\tilde{\mu}^{-1}(s_o)) = n(\mu_1^{-1}(s_o)) + n(\mu_2^{-1}(\frac{1}{2}s_o)) - 1 = 1 + n_o - 1 = n_o$. If $s_o > \mu_1(X_1)$, then $n(\tilde{\mu}^{-1}(s_o)) = n(\mu_2^{-1}(\frac{1}{2}s_o)) = n_o$. Since $n_o \geq 2$, we have $n_o \neq 1$ and $2n_o - 1 > n_o$, so $\tilde{\mu}^{-1}(s_o)$ cannot be homeomorphic to $\bar{\mu}^{-1}(t)$ for any t. Therefore, X admits essentially different Whitney maps.

Lemma 31. Let X *be a one-dimensional polyhedron,* $B \in C(X)$, *and* $p \in B$ *such that the order of* p *is greater than or equal to 3. Let* $\mu(B) = t$ *where* $0 < t < \mu(X)$. *Then the order of* B *in* $\mu^{-1}(t)$ *is greater than or equal to 3.*

Proof. We consider 2 cases.

Case I: Suppose X is a tree. Then X can be expressed as a union of continua $\cup_{i=1}^{3} A_i$ with $A_i \cap A_j = \{p\}$ if $i \neq j$. For i = 1, 2, 3 there exists a segment $\phi_i: I \to C(X)$ from {p} to A_i containing the point $B \cap A_i$. Using these segments it is easy to construct three arcs in $\mu^{-1}(t)$ such that the intersection of any two of them is the point B. Hence B is of order 3 or more in $\mu^{-1}(t)$.

Case II: Suppose X is not a tree. Then there exists a tree Y and a finite collection $\{(p_i,p_i')\}$ of pairs of endpoints of Y such that if each p_i is identified with p_i' then the resulting space is homeomorphic to X. Let $f: Y \to X$ be this identification map. Assume also that the pairs of points (p_i,p_i') are chosen so that $f^{-1}(B)$ is a continuum.

By Proposition 2.1 of [8], f induces a map $\hat{f}: C(Y) \to C(X)$ such that $\hat{f}$ embeds $C(Y)\backslash\hat{Y}$ into $C(X)\backslash\hat{X}$ and $\mu\circ\hat{f}$ is a Whitney map on C(Y). By Case I, $f^{-1}(B)$ is of order 3 or more in $(\mu\circ\hat{f})^{-1}(t)$. Hence $B = \hat{f}(f^{-1}(B))$ has order at least 3 in $\mu^{-1}(t)$.

Proposition 32. Let X *be a one-dimensional polyhedron which is not an arc or a circle. Then* X *admits essentially different Whitney maps.*

Proof. We may express X as a union of continua $\bigcup_{i=1}^{k} A_i$, $k \geq 2$ such that (1) each A_i is an arc or an arc with endpoints identified, (2) if $i \neq j$ then $A_i \cap A_j$ is contained in the set of endpoints of A_i and A_j, and (3) no endpoint has order 2. Note that each A_i has at least one endpoint of order 3 or more.

Let μ be a Whitney map on C(X). If for some i we have $0 < t < \mu(A_i)$, then $C(A_i,t)$ is a free arc in $\mu^{-1}(t)$. Suppose $\mu(A_i) < t < \mu(X)$. Then if $B \in \mu^{-1}(t)$ and $B \cap A_i \neq \emptyset$, B must contain an endpoint of A_i having order at least 3. Thus, by Lemma 31, B is of order 3 or more in $\mu^{-1}(t)$, so B cannot be in the interior of a free arc in $\mu^{-1}(t)$. We use this information to construct essentially different Whitney maps on C(X).

Let μ_1 be a Whitney map on C(X) such that $\mu_1(A_i) = s_o$ for $i = 1,\cdots,k-1$, and $\mu_1(A_k) = s_1$ where $s_1 > s_o$. Choose s so that $s_o < s < s_1$. By the argument above, $\mu_1^{-1}(s)$ contains exactly one free arc. Let μ_2 be a Whitney map on C(X) such that $\mu_2(A_i) = t_o$ for i = 1,•••,k. Then for $t < t_o$, $\mu_2^{-1}(t)$ contains exactly k free arcs and for $t \geq t_o$, $\mu_2^{-1}(t)$ contains no free arcs. Hence no $\mu_2^{-1}(t)$ is homeomorphic to $\mu_1^{-1}(s)$.

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