TOPOLOGY PROCEEDINGS

Volume 1, 1976

Pages 173–179

http://topology.auburn.edu/tp/

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by

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Topology Proceedings

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ISSN:	0146-4124

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1. Introduction

The principal purpose of this paper is to prove the following theorem about the structure of small normal F-spaces, and to derive some corollaries of it. (We call a space X "small" if $|C^*(X)| = 2^\omega$; explanations of other terminology and notation appear below.)

1.1 *Theorem. Assume the continuum hypothesis. Let* X *be a normal* F*-space such that* $|C^*(X)| = 2^\omega$. *Then:*

(a) If X *is countably compact then* X *is compact.*

(b) If X *is locally compact then* X *is* σ*-compact.*

All hypothesized topological spaces are assumed to be completely regular and Hausdorff. Throughout this paper we shall use the notation and terminology of the Gillman-Jerison text [4] without further comment. We shall however remind the reader of the definition of a few of the concepts that appear below.

A topological space is called an *F-space* if its cozero-sets are C*-embedded. A space is *extremally disconnected* if each of its open sets has an open closure. Each extremally disconnected space is an F-space; see 14N.4 of [4]. A space X is called *weakly Lindelöf* if given an open cover $\mathcal{U}$ of X, there is a countable subcollection $\mathcal{U}'$ of $\mathcal{U}$ such that $\bigcup\{U: U \in \mathcal{U}'\}$ is dense in X. The Stone-Čech compactification of X is denoted by βX; the Hewitt realcompactification is denoted by υX. The cardinality of a set S is denoted by $|S|$. The countable discrete space is denoted by

[1]This research was partially supported by Grant No. A7592 from the National Research Council of Canada.

[2]A.M.S. 1970 Subject Classification: Primary 54G05, 54D15 Secondary 54D60. Key words and phrases: F-space, normal space, extremally disconnected, countably compact, absolute, ω-bounded.

$\underline{N}$. The set of bounded real-valued continuous functions on a space X is denoted by $C^*(X)$. If we use the continuum hypothesis ($2^\omega = \omega_1$) in the proof of a theorem we indicate this by writing "[CH]" before the statement of the theorem.

The following two theorems will be used in the sequel. The first appears as part of Theorem 4.6 of [2].

1.2 *Theorem* [CH]. *Let* Y *be a locally compact* σ*-compact non-compact space such that* $|C^*(Y)| = 2^\omega$. *Then* $\beta Y - Y$ *contains no proper dense* C^**-embedded subspace, and an open subspace of* $\beta Y - Y$ *is* C^**-embedded in* $\beta Y - Y$ *iff it is a cozero-set of* $\beta Y - Y$.

The following appears as the non-trivial part of Theorem 2.2 of [13].

1.3 *Theorem* [CH]. *Let* K *be a compact* F*-space such that* $|C^*(K)| = 2^\omega$. *If* X *is a* C^**-embedded subspace of* K *then* X *is weakly Lindelöf.*

2. Proof of Theorem 1.1 and Its Corollaries

Proof of Theorem 1.1. As X is an F-space, so is βX (see 14.25 of [4]). Since X is C^*-embedded in βX, by 1.3 X is weakly Lindelöf.

To prove 1.1(a), suppose X is not compact. Choose $p \in \beta X - X$ and write $\beta X - \{p\}$ as a union of cozero-sets of βX. As X is weakly Lindelöf there are countably many of these cozero-sets whose union, when intersected with X, yields a dense subspace of X. Let V denote this union. Then V is a dense cozero-set of βX not containing p. As V is C^*-embedded in the F-space βX, it follows that $\beta X = \beta V$. As V satisfies the hypotheses imposed on Y in 1.2, by 1.2 $\beta X - V$ has no proper dense C^*-embedded subset. We now show that $X - V$ is a proper dense C^*-embedded subset of $\beta X - V$, thus obtaining a contradiction.

The closed subspace $X - V$ of the normal space X is C*-embedded in X, and hence in βX, and hence in $\beta X - V$. Furthermore $p \in (\beta X - V) - (X - V)$. To show that $X - V$ is dense in $\beta X - V$, let A be an open subset of βX meeting $\beta X - V$. Since V is a cozero-set of βX it is an F_σ-set, so $A - V$ is a non-empty G_δ-set of βX. As X is countably compact, $(A - V) \cap X \neq \emptyset$ (see 8A.4 and 8.8 of [4]); thus $X - V$ is dense in $\beta X - V$. This contradiction shows that $\beta X - X$ could not have been non-empty, so X is compact.

To prove 1.1(b) first note that X is open in βX since X is locally compact (see 3.15(d) of [4]). Write X as a union of cozero-sets of βX; since X is weakly Lindelöf there is a countable subfamily of these cozero-sets whose union U is a dense cozero-set of βX and is thus C*-embedded in βX. Thus $U \subset X \subset \beta X = \beta U$, and U satisfies the hypotheses imposed on Y in 1.2. Thus by 1.2 any open C*-embedded subspace of $\beta X - U$ is a cozero-set of $\beta X - U$, and hence is σ-compact (as $\beta X - U$ is compact). But $X - U$ is open in $\beta X - U$ as X is open in βX, and $X - U$ is C*-embedded in $\beta X - U$ since X is normal and its closed subspace $X - U$ is therefore C*-embedded in X. Thus X is the union of two σ-compact spaces and hence it is σ-compact.

We next derive some corollaries to Theorem 1.1. There has been some interest in determining whether a product of normal countably compact spaces need to be countably compact; see for example Problem B15 of [9]. Corollary 2.2 gives an affirmative answer for a special case. Recall that a space is *ω-bounded* if its countable subsets are relatively compact.

2.1 *Corollary* [CH]. *A normal countably compact* F-*space is ω-bounded. Hence a product of arbitrarily many normal countably compact* F-*spaces is countably compact.*

Proof. Let S be a countable subset of the normal countably compact F-space X. Then $c\ell_X S$ is separable, normal and countably compact; as it is C*-embedded in X, by 14.26 of [4] it is an F-space. Obviously $|C^*(c\ell_X S)| = 2^\omega$ so by 1.1(a) $c\ell_X S$ is compact. The remainder of the corollary follows from the fact that products of ω-bounded spaces are ω-bounded, and ω-bounded spaces are countably compact.

We next use 1.1(a) to prove a generalization of 1.1(a).

2.2 *Corollary* [CH]. *Let* X *be a normal* F-*space such that* $|C^*(X)| = 2^\omega$. *Then* υX *is not locally compact at any point of* $\upsilon X - X$.

Proof. Suppose that $p \in \upsilon X - X$, V is open in υX, $p \in V$, and $c\ell_{\upsilon X} V$ is compact. By 4.1 of [1] $X \cap c\ell_{\upsilon X} V$ is pseudocompact. It is also normal so by 3D.2 of [4] it is countably compact. As $X \cap c\ell_{\upsilon X} V$ is C*-embedded in X, it is an F-space by 14.25 of [4] and $|C^*(X \cap c\ell_{\upsilon X} V)| = 2^\omega$. Hence by 1.1 $X \cap c\ell_{\upsilon X} V$ is compact. But $c\ell_{\upsilon X} V = c\ell_{\upsilon X}(X \cap c\ell_{\upsilon X} V)$ and $p \in c\ell_{\upsilon X} V - X$. From this contradiction the corollary follows.

Recall that the *absolute* E(X) of a space X is (the unique) extremally disconnected space that can be mapped onto X by a map that is perfect and irreducible (i.e. the map takes proper closed subsets of E(X) to proper closed subsets of X). See [10] and [12] for details. The proof of the following well-known "folk lemma" is straightforward and is not included.

2.3 *Lemma. If* P *is countable compactness, or* ω-*boundedness, or separability, or local compactness, then a space* X *has property* P *iff* E(X) *has* P.

There has recently been some interest in determining conditions under which E(X) is normal. Hence the following corollary is of interest.

2.4 *Corollary* [CH]. *Assume that* E(X) *is normal.*

(a) If X *is countably compact then* X *is* ω*-bounded.*

(b) If X *is separable and locally compact then* X *is* σ*-compact.*

(c) If X *is separable and* υX *is locally compact then* X *is* σ*-compact.*

(d) If X *is separable and countably compact then* X *is compact.*

Proof. (a) this follows immediately from 2.1 and 2.3.

(b) This follows from 1.1(b) and 2.3.

(c) As υX is locally compact, by [8], page 237, or [12], Theorem 2.10, $E(\upsilon X) = \upsilon E(X)$. Hence by 2.3 $\upsilon E(X)$ is locally compact. By 2.3 E(X) is separable and so $|C^*(E(X))| = 2^{\omega}$. Thus by 2.2 it follows that $\upsilon E(X) = E(X)$. Thus $E(\upsilon X) = E(X)$ so $\upsilon X = X$, i.e. X is locally compact. The result now follows from (b).

(d) This follows immediately from (a) or (c).

Conditions on X equivalent to the local compactness of υX may be found in Harris [5].

3. Examples and Questions

The following examples are designed to show that most of the hypotheses of Theorem 1.1 and its corollaries are necessary to their proofs.

3.1 *Examples.* The space of countable ordinals (with the order topology) is a non-compact space satisfying all the hypotheses of 1.1 except that it is not an F-space. Under assumption of the continuum hypothesis, the space $\gamma\underline{N} - \{\omega_1\}$ constructed by Franklin and Rajogapalan in [3] is a separable non-compact space satisfying all the hypotheses of 1.1 except that it is not an F-space.

3.2 *Example* [CH]. Let $p \in \beta\underline{N} - \underline{N}$. Then $\beta\underline{N} - \{p\}$ is a separable non-compact space satisfying all the hypotheses of 1.1 except that it is not normal.

3.3 *Example.* Examples abound of normal non-compact F-spaces X such that $|C^*(X)| = 2^\omega$; a non-compact cozero set of $\beta\underline{N} - \underline{N}$ is such a space.

3.4 *Example.* Using the set-theoretic hypothesis ♣, which is known to be consistent with the continuum hypothesis (see page 32 of [9]), M. Wage has recently constructed a separable normal extremally disconnected space that is not realcompact (see [11]). This shows that the hypotheses on X in 2.2 do not imply that X must be realcompact. It also shows that local compactness cannot be dropped from the hypothesis of 1.1(b), since σ-compact spaces are realcompact.

3.5 *Example.* Kunen and Parsons [7] have recently shown that if $\aleph$ is a weakly compact cardinal, and if E denotes the subspace of $\beta\aleph$ (where $\aleph$ is given the discrete topology) consisting of those ultrafilters that contain a set of cardinality less than $\aleph$, then E is a normal, countably compact, non-compact extremally disconnected space. Hence the assumption that $|C^*(X)| = 2^\omega$ cannot be dropped from Theorem 1.1. We do not know whether it can be replaced by some significantly weaker assumption.

We conclude with two open questions.

3.6 *Question.* Is [CH] necessary to prove 1.1? Does Theorem 1.1 hold without any special set-theoretic assumptions?

3.7 *Question.* Is there a "real" example of an extremally disconnected locally compact normal space that is not paracompact? Theorem 1.1(b) says that if X is such a space then $|C^*(X)| > 2^\omega$.

Example 3.5 says that if one assumes the existence of weakly compact cardinals then such spaces exist. Kunen [6] has recently constructed a "real" normal extremally disconnected subspace of $\beta\omega_1$ (where ω_1 has the discrete topology) that is not collectionwise Hausdorff, and thus not paracompact; however, his example is not locally compact. (By "real" we mean that no special set-theoretic hypotheses are used in the construction.)

If $2^{\omega} = 2^{\omega_1}$ then the discrete space of cardinality ω_1 becomes a counterexample to 1.1(b). Hence the assumption of the continuum hypothesis cannot be dropped from 1.1(b). I do not know if it can be replaced by the assumption that $2^{\omega} < 2^{\omega_1}$.

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