# TOPOLOGY PROCEEDINGS

**Volume 1, 1976**

Pages 173–179

**http://topology.auburn.edu/tp/**


# THE STRUCTURE OF SMALL NORMAL $F$-SPACES


by

R. Grant Woods



**Topology Proceedings**

| | |
|---|---|
| **Web:** | http://topology.auburn.edu/tp/ |
| **Mail:** | Topology Proceedings<br>Department of Mathematics & Statistics<br>Auburn University, Alabama 36849, USA |
| **E-mail:** | topolog@auburn.edu |
| **ISSN:** | 0146-4124 |



COPYRIGHT © by Topology Proceedings. All rights reserved.

# THE STRUCTURE OF SMALL NORMAL F-SPACES

R. Grant Woods [1, 2]

## 1. Introduction

The principal purpose of this paper is to prove the following theorem about the structure of small normal F-spaces, and to derive some corollaries of it. (We call a space X "small" if $|C^*(X)| = 2^{\omega}$; explanations of other terminology and notation appear below.)

1.1 *Theorem. Assume the continuum hypothesis. Let* X *be a normal* F*-space such that* $|C^*(X)| = 2^{\omega}$. *Then:*

*(a) If* X *is countably compact then* X *is compact.*

*(b) If* X *is locally compact then* X *is* $\sigma$*-compact.*

All hypothesized topological spaces are assumed to be completely regular and Hausdorff. Throughout this paper we shall use the notation and terminology of the Gillman-Jerison text [4] without further comment. We shall however remind the reader of the definition of a few of the concepts that appear below.

A topological space is called an *F-space* if its cozero-sets are C*-embedded. A space is *extremally disconnected* if each of its open sets has an open closure. Each extremally disconnected space is an F-space; see 14N.4 of [4]. A space X is called *weakly Lindelöf* if given an open cover $\mathfrak{U}$ of X, there is a countable subcollection $\mathfrak{U}'$ of $\mathfrak{U}$ such that $\bigcup\{U: U \in \mathfrak{U}'\}$ is dense in X. The Stone-Čech compactification of X is denoted by $\beta X$; the Hewitt realcompactification is denoted by $\upsilon X$. The cardinality of a set S is denoted by $|S|$. The countable discrete space is denoted by

[1] This research was partially supported by Grant No. A7592 from the National Research Council of Canada.

[2] A.M.S. 1970 Subject Classification: Primary 54G05, 54D15 Secondary 54D60. Key words and phrases: F-space, normal space, extremally disconnected, countably compact, absolute, $\omega$-bounded.

N. The set of bounded real-valued continuous functions on a space X is denoted by $C^*(X)$. If we use the continuum hypothesis ($2^{\omega} = \omega_1$) in the proof of a theorem we indicate this by writing "[CH]" before the statement of the theorem.

The following two theorems will be used in the sequel. The first appears as part of Theorem 4.6 of [2].

1.2 *Theorem* [CH]. *Let* Y *be a locally compact* $\sigma$*-compact non-compact space such that* $|C^*(Y)| = 2^{\omega}$. *Then* $\beta Y - Y$ *contains no proper dense* $C^*$*-embedded subspace, and an open subspace of* $\beta Y - Y$ *is* $C^*$*-embedded in* $\beta Y - Y$ *iff it is a cozero-set of* $\beta Y - Y$.

The following appears as the non-trivial part of Theorem 2.2 of [13].

1.3 *Theorem* [CH]. *Let* K *be a compact* F*-space such that* $|C^*(K)| = 2^{\omega}$. *If* X *is a* $C^*$*-embedded subspace of* K *then* X *is weakly Lindelöf.*

**2. Proof of Theorem 1.1 and Its Corollaries**

*Proof of Theorem 1.1.* As X is an F-space, so is $\beta X$ (see 14.25 of [4]). Since X is $C^*$-embedded in $\beta X$, by 1.3 X is weakly Lindelöf.

To prove 1.1(a), suppose X is not compact. Choose $p \in \beta X - X$ and write $\beta X - \{p\}$ as a union of cozero-sets of $\beta X$. As X is weakly Lindelöf there are countably many of these cozero-sets whose union, when intersected with X, yields a dense subspace of X. Let V denote this union. Then V is a dense cozero-set of $\beta X$ not containing p. As V is $C^*$-embedded in the F-space $\beta X$, it follows that $\beta X = \beta V$. As V satisfies the hypotheses imposed on Y in 1.2, by 1.2 $\beta X - V$ has no proper dense $C^*$-embedded subset. We now show that $X - V$ is a proper dense $C^*$-embedded subset of $\beta X - V$, thus obtaining a contradiction.

The closed subspace X - V of the normal space X is C*-embedded in X, and hence in βX, and hence in βX - V. Furthermore $p \in (\beta X - V) - (X - V)$. To show that X - V is dense in βX - V, let A be an open subset of βX meeting βX - V. Since V is a cozero-set of βX it is an $F_\sigma$-set, so A - V is a non-empty $G_\delta$-set of βX. As X is countably compact, $(A - V) \cap X \neq \emptyset$ (see 8A.4 and 8.8 of [4]); thus X - V is dense in βX - V. This contradiction shows that βX - X could not have been non-empty, so X is compact.

To prove 1.1(b) first note that X is open in βX since X is locally compact (see 3.15(d) of [4]). Write X as a union of cozero-sets of βX; since X is weakly Lindelöf there is a countable subfamily of these cozero-sets whose union U is a dense cozero-set of βX and is thus C*-embedded in βX. Thus $U \subset X \subset \beta X = \beta U$, and U satisfies the hypotheses imposed on Y in 1.2. Thus by 1.2 any open C*-embedded subspace of βX - U is a cozero-set of βX - U, and hence is σ-compact (as βX - U is compact). But X - U is open in βX - U as X is open in βX, and X - U is C*-embedded in βX - U since X is normal and its closed subspace X - U is therefore C*-embedded in X. Thus X is the union of two σ-compact spaces and hence it is σ-compact.

We next derive some corollaries to Theorem 1.1. There has been some interest in determining whether a product of normal countably compact spaces need to be countably compact; see for example Problem B15 of [9]. Corollary 2.2 gives an affirmative answer for a special case. Recall that a space is *ω-bounded* if its countable subsets are relatively compact.

2.1 *Corollary* [CH]. *A normal countably compact* F*-space is ω-bounded. Hence a product of arbitrarily many normal countably compact* F*-spaces is countably compact.*

*Proof.* Let S be a countable subset of the normal countably

compact F-space X. Then $c\ell_X S$ is separable, normal and countably compact; as it is C*-embedded in X, by 14.26 of [4] it is an F-space. Obviously $|C^*(c\ell_X S)| = 2^\omega$ so by 1.1(a) $c\ell_X S$ is compact. The remainder of the corollary follows from the fact that products of $\omega$-bounded spaces are $\omega$-bounded, and $\omega$-bounded spaces are countably compact.

We next use 1.1(a) to prove a generalization of 1.1(a).

2.2 *Corollary* [CH]. *Let* X *be a normal* F*-space such that* $|C^*(X)| = 2^\omega$. *Then* $\upsilon X$ *is not locally compact at any point of* $\upsilon X - X$.

*Proof.* Suppose that $p \in \upsilon X - X$, V is open in $\upsilon X$, $p \in V$, and $c\ell_{\upsilon X} V$ is compact. By 4.1 of [1] $X \cap c\ell_{\upsilon X} V$ is pseudocompact. It is also normal so by 3D.2 of [4] it is countably compact. As $X \cap c\ell_{\upsilon X} V$ is C*-embedded in X, it is an F-space by 14.25 of [4] and $|C^*(X \cap c\ell_{\upsilon X} V)| = 2^\omega$. Hence by 1.1 $X \cap c\ell_{\upsilon X} V$ is compact. But $c\ell_{\upsilon X} V = c\ell_{\upsilon X}(X \cap c\ell_{\upsilon X} V)$ and $p \in c\ell_{\upsilon X} V - X$. From this contradiction the corollary follows.

Recall that the *absolute* E(X) of a space X is (the unique) extremally disconnected space that can be mapped onto X by a map that is perfect and irreducible (i.e. the map takes proper closed subsets of E(X) to proper closed subsets of X). See [10] and [12] for details. The proof of the following well-known "folk lemma" is straightforward and is not included.

2.3 *Lemma. If* P *is countable compactness, or* $\omega$*-boundedness, or separability, or local compactness, then a space* X *has property* P *iff* E(X) *has* P.

There has recently been some interest in determining conditions under which E(X) is normal. Hence the following corollary is of interest.

2.4 *Corollary* [CH]. *Assume that* E(X) *is normal.*

*(a) If* X *is countably compact then* X *is* ω*-bounded.*

*(b) If* X *is separable and locally compact then* X *is* σ*-compact.*

*(c) If* X *is separable and* υX *is locally compact then* X *is* σ*-compact.*

*(d) If* X *is separable and countably compact then* X *is compact.*

*Proof.* (a) this follows immediately from 2.1 and 2.3.

(b) This follows from 1.1(b) and 2.3.

(c) As υX is locally compact, by [8], page 237, or [12], Theorem 2.10, E(υX) = υE(X). Hence by 2.3 υE(X) is locally compact. By 2.3 E(X) is separable and so $|C^*(E(X))| = 2^{\omega}$. Thus by 2.2 it follows that υE(X) = E(X). Thus E(υX) = E(X) so υX = X, i.e. X is locally compact. The result now follows from (b).

(d) This follows immediately from (a) or (c).

Conditions on X equivalent to the local compactness of υX may be found in Harris [5].

**3. Examples and Questions**

The following examples are designed to show that most of the hypotheses of Theorem 1.1 and its corollaries are necessary to their proofs.

3.1 *Examples.* The space of countable ordinals (with the order topology) is a non-compact space satisfying all the hypotheses of 1.1 except that it is not an F-space. Under assumption of the continuum hypothesis, the space $\gamma\underline{N} - \{\omega_1\}$ constructed by Franklin and Rajogapalan in [3] is a separable non-compact space satisfying all the hypotheses of 1.1 except that it is not an F-space.

3.2 *Example* [CH]. Let $p \in \beta\underline{N} - \underline{N}$. Then $\beta\underline{N} - \{p\}$ is a separable non-compact space satisfying all the hypotheses of 1.1 except that it is not normal.

3.3 *Example.* Examples abound of normal non-compact F-spaces X such that $|C^*(X)| = 2^{\omega}$; a non-compact cozero set of $\beta\underline{N} - \underline{N}$ is such a space.

3.4 *Example.* Using the set-theoretic hypothesis $\clubsuit$, which is known to be consistent with the continuum hypothesis (see page 32 of [9]), M. Wage has recently constructed a separable normal extremally disconnected space that is not realcompact (see [11]). This shows that the hypotheses on X in 2.2 do not imply that X must be realcompact. It also shows that local compactness cannot be dropped from the hypothesis of 1.1(b), since $\sigma$-compact spaces are realcompact.

3.5 *Example.* Kunen and Parsons [7] have recently shown that if $\aleph$ is a weakly compact cardinal, and if E denotes the subspace of $\beta\aleph$ (where $\aleph$ is given the discrete topology) consisting of those ultrafilters that contain a set of cardinality less than $\aleph$, then E is a normal, countably compact, non-compact extremally disconnected space. Hence the assumption that $|C^*(X)| = 2^{\omega}$ cannot be dropped from Theorem 1.1. We do not know whether it can be replaced by some significantly weaker assumption.

We conclude with two open questions.

3.6 *Question.* Is [CH] necessary to prove 1.1? Does Theorem 1.1 hold without any special set-theoretic assumptions?

3.7 *Question.* Is there a "real" example of an extremally disconnected locally compact normal space that is not paracompact? Theorem 1.1(b) says that if X is such a space then $|C^*(X)| > 2^{\omega}$.

Example 3.5 says that if one assumes the existence of weakly compact cardinals then such spaces exist. Kunen [6] has recently constructed a "real" normal extremally disconnected subspace of $\beta\omega_1$ (where $\omega_1$ has the discrete topology) that is not collectionwise Hausdorff, and thus not paracompact; however, his example is not locally compact. (By "real" we mean that no special set-theoretic hypotheses are used in the construction.)

If $2^{\omega} = 2^{\omega_1}$ then the discrete space of cardinality $\omega_1$ becomes a counterexample to 1.1(b). Hence the assumption of the continuum hypothesis cannot be dropped from 1.1(b). I do not know if it can be replaced by the assumption that $2^{\omega} < 2^{\omega_1}$.

**Bibliography**


1. W. W. Comfort, *On the Hewitt realcompactification of a product space*, Trans. Amer. Math. Soc. 131 (1968), 107-118.
2. N. J. Fine and L. Gillman, *Extensions of continuous functions in* $\beta\underline{N}$, Bull. Amer. Math. Soc. 66 (1960), 376-381.
3. S. Franklin and M. Rajagopolan, *Some examples in topology*, Trans. Amer. Math. Soc. 155 (1971), 305-314.
4. L. Gillman and M. Jerison, *Rings of continuous functions*, Princeton, N.J., Van Nostrand, 1960.
5. D. Harris, *The local compactness of* $\upsilon X$, Pacific J. Math 50 (1974), 469-476.
6. K. Kunen, private communication.
7. K. Kunen and L. Parsons, to appear.
8. J. Mack and D. Johnson, *The Dedekind completion of* C(X), Pacific J. Math 20 (1967), 231-243.
9. M. E. Rudin, *Lectures in set-theoretic topology*, C.B.M.S. Regional Conference Series in Mathematics No. 23 (1974).
10. D. P. Strauss, *Extremally disconnected spaces*, Proc. Amer. Math. Soc. 18 (1967), 305-309.
11. M. Wage, *Extremally disconnected S-spaces*, to appear in these proceedings.
12. R. G. Woods, *Ideals of pseudocompact regular closed sets and absolutes of Hewitt realcompactifications*, General Topology and Its Applic. 2 (1972), 315-331.
13. _____, *Characterizations of some* C**-embedded subspaces of* $\beta\underline{N}$, to appear in Pacific J. Math.


The University of Manitoba, Winnipeg, Canada