TOPOLOGY PROCEEDINGS

Volume 1, 1976

Pages 181–185

http://topology.auburn.edu/tp/

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Topology Proceedings

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ISSN:	0146-4124

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EXTREMALLY DISCONNECTED S-SPACES

Michael L. Wage

Recently there has been much work done on S-spaces and L-spaces (see [K], [JKR], and [R] for example). Examples of S-spaces have previously been constructed under diamond by Ostaszewski and under CH by Hajnal, Juhász, Kunen, and Rudin, but no absolute example has been constructed. In this note we give, under club, an example of an extremally disconnected S-space and prove that such spaces are normal and, in many cases, perfect. The addition of extremal disconnectivity not only forces S-spaces to be nice, but also seems to simplify their construction.

Definitions and notation. A space is called *extremally disconnected* if the closure of every open set is open. This is equivalent to saying that disjoint open sets have disjoint closures X is called an *S-space* if X is regular, hereditarily separable, but not Lindelöf. X is a *strong S-space* if each finite power of X is an S-space. We use the term *perfect* to mean that each closed subset is a G_δ. A^- and A^c denote, respectively, the closure and complement of A. Two subsets of a space, A and B, are called *separated* if $A^- \cap B = \phi = A \cap B^-$. The only set theoretic assumption used in this note is club, an axiom that is independent of and consistent with the usual axioms for set theory. Let Λ be the set of all limit points in ω_1. Club states that there exists a collection $\{S_\lambda \mid \lambda \in \Lambda\}$ such that

a) for all $\lambda \in \Lambda$, S_λ is a cofinal subset of λ that has finite intersection with each ordinal less than λ, and

b) if C is an uncountable subset of ω_1, then there exists a λ such that $S_\lambda \subset C$.

Theorem 1. Each extremally disconnected S*-space is hereditarily normal.*

Proof. Let H and K be separated (see the definition above) subsets of an extremally disconnected S-space X. Since X is hereditarily separable, there exist countable sets $D \subset H$ and $E \subset K$ such that D is dense in H and E is dense in K. D and E are countable separated subsets of a regular space and hence can be separated by disjoint open sets, $U \supset D$ and $V \supset E$. The extremal disconnectivity of X implies that U^- and V^- are two disjoint open sets that separate H from K. It follows that each pair of separated sets can be separated by disjoint open sets, and thus X is hereditarily normal.

Example. Although each regular hereditarily separable extremally disconnected space is normal, it is not true that each Hausdorff hereditarily separable extremally disconnected space is regular. An easy example is the following: Let $N = \{(n,m) \mid n,m \leq \omega\} - \{(n,\omega) \mid n < \omega\}$ and fix a free ultrafilter, F, on ω. Declare each point of $\{(n,m) \mid n,m < \omega\}$ to be open. For each $f \in F$ and $m < \omega$, a basic open set around the point (ω,m) is defined to be $\{(\omega,m)\} \cup \{(n,m) \mid n \in f\}$. For each $g \in F$ and each sequence $\{f_m \mid m < \omega\} \subset F$, we define $\{(\omega,\omega)\} \cup \{(n,m) \mid n \in f_m, m \in g\}$ to be a basic open neighborhood of the point (ω,ω).

Lemma. Let X *be an extremally disconnected* S*-space. If* H *is closed in* X, U *is open in* X, *and* $H \cap U$ *is dense in* H, *then* $H \cup U$ *is an open set.*

Proof. Suppose to the contrary that X, H, and U are as in the hypothesis of the lemma but $H \cup U$ is not open. Then there is a point $h \in H$ such that h is in the closure of $X-(H \cup U)$. Let D be a countable dense subset of $H \cap U$ and E be a countable dense subset of $X-(H \cup U)$. Notice that D and E are countable separated subsets of X and thus can be separated by disjoint

open sets. But h is in the closure of both D and E, and hence is in the closure of the disjoint open sets that separate them. This contradicts the extremal disconnectivity of X and finishes the proof of the lemma.

Theorem 2. If X *is an extremally disconnected* S*-space in which each countable set is a* G_δ*, then* X *is perfect.*

Proof. Let H be a closed subset of X and let D be a countable dense subset of H. By assumption, there exist open sets, U_n, $n < \omega$, such that $D = \bigcap \{U_n | n<\omega\}$. It follows from the lemma that for each n, $U_n \cup H$ is an open set. Then $H = \bigcap \{U_n \cup H | n<\omega\}$ and hence is a G_δ.

This theorem is especially interesting since most of the spaces constructed using Ostaszewski's technique of building spaces by induction are locally countable, and hence have countable subsets being G_δ.

The following related theorem has been proved by A. Berner and the author in [W].

Theorem 3. There is no extremally disconnected hereditarily separable Dowker space.

Example. Club implies there exists an extremally disconnected S-space. Let $M = \omega_1$. We will inductively define topologies, τ_λ, on initial segments, λ, of M. Recall that $\{S_\lambda | \lambda \in \Lambda\}$ is the sequence associated with club. Let τ_ω be the discrete topology on ω. Suppose $\lambda \in \Lambda$ and we have defined τ_ξ on ξ for each limit ordinal $\xi < \lambda$ such that

1) τ_ξ is T_3 and extremally disconnected,

2) $\tau_\gamma = \tau_\xi \cap P(\gamma)$ and $\eta \in \tau_\xi$ for all $\eta, \gamma < \xi$ such that γ is a limit, and

3) $\eta \in cl_{\tau_\xi}(S_\gamma)$ for all $\gamma \leq \eta < \xi$ such that γ is a limit.

We must show how to define τ_λ on λ so that (1)-(3) hold for

all limits $\xi \leq \lambda$.

If λ is a limit of limits, let τ_λ be the topology generated by the base $\bigcup_{\xi<\lambda}\tau_\xi$. Conditions (2) and (3) above are then easily verified for each limit $\xi \leq \lambda$ as is extremal disconnectivity. Regularity is a little trickier. Suppose $\alpha \in U$, an open set in (λ,τ_λ). To show that (λ,τ_λ) is regular, we will find a $V \subset U$ that is clopen in (λ,τ_λ) and contains α. Without loss of generality, $U \subset \alpha+1$ by condition (2). Let $\xi = \alpha+\omega$. Then since τ_ξ is regular and extremally disconnected, there exists a V, clopen in τ_ξ, with $\alpha \in V \subset U$. V is open in τ_λ, but we must show it is clopen in τ_λ to complete the proof of regularity. Consider ξ-V. Since V is closed in τ_ξ, ξ-V is open in τ_ξ, and hence, by condition (2), open in τ_λ. But ξ-V contains all but finitely many points of S_ξ (since $\xi = \alpha+\omega$ so that S_ξ, by definition, has finite intersection with $\alpha+1$, and hence with V) and by condition (3), $\lambda-\xi \subset cl_{\tau_\lambda}(S_\xi)$, so $cl_{\tau_\lambda}(\xi-V) = \lambda-V$. Since in an extremally disconnected space, disjoint open sets have disjoint closures, V must be closed in τ_λ, and the limit of limits case is completed.

Now suppose $\lambda = \gamma+\omega$ for some limit γ. We must define neighborhoods of the points $\gamma+n$ for each $n < \omega$. Partition S_γ into ω infinite pieces, W_n, and fix a free ultrafilter F_n on W_n for each $n < \omega$. A basic neighborhood of $\gamma+n$ is defined to be any set of the form $\{\gamma+n\} \cup U$ where $U \in \tau_\gamma$ and U contains some member of F_n. Let these basic neighborhoods, together with τ_γ, be a base for τ_λ. With this definition it is easy to check (1)-(3) for each limit $\xi \leq \lambda$. Regularity follows from the fact that τ_γ is normal (it is Lindelöf and regular) and S_γ is closed discrete in τ_γ (by condition (2)).

Finally, we let $\tau = \tau_{\omega_1}$ be the topology on M. We have already shown that M is extremally disconnected and regular and condition (3) guarantees that M is an S-space. Together,

Theorems 1 and 2 show that M is perfect and hereditarily normal.

Remarks. A slight modification of M shows that not every extremally disconnected S-space is perfect: Let $M^* = M \cup \{\omega_1\}$. Topologize M^* by giving M the usual topology, τ, and letting the basic neighborhoods of the point ω_1 be $\{\omega_1\} \cup U$ where U is any co-countable member of τ. Then M^* is an extremally disconnected S-space but is not perfect since the point ω_1 is not a G_δ.

The space M is not realcompact. Ken Kunen noticed that a realcompact extremally disconnected S-space can be constructed by using the above technique if the construction is done on a Souslin tree. Kunen also noticed that M^2 is not hereditarily separable and hence M is not a strong S-space. The existence of an extremally disconnected strong S-space is still an open question.

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