TOPOLOGY PROCEEDINGS
Volume 1, 1976
Pages 187–189
http://topology.auburn.edu/tp/

WHITNEY CONTINUUM IN HYPERSPACE

by

A.Y. W. Lau

Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: 0146-4124

WHITNEY CONTINUUM IN HYPERSPACE

A. Y. W. Lau

A continuum means a compact connected metrizable space and $C(X)$ is the hyperspace of subcontinua of X with the Vietoris topology. A continuous function $\mu:C(X) \to [0,1]$ is a Whitney function if $\mu(X) = 1$, $\mu(\{p\}) = 0$ for each $p \in X$ and $\mu(A) < \mu(B)$ if A is a proper subset of B (see [2] and [3]). Let $\hat{X}$ be the set of singletons of X and $D(X) = C(X)/\hat{X}$ (i.e., decomposition of $C(X)$ into elements and $\hat{X}$). The reduced Alexander cohomology H^p is employed (see [7]) and X is acyclic if $H^p(X) = 0$ for each p. If $A \subseteq B$ and $e \in H^p(B)$, then $e|A = i^*(e)$ where i is the inclusion map.

Define $H \leq K$ in $D(X)$ if $H \subseteq K$ or $H = \hat{X}$. If $\Sigma \subseteq D(X)$, then $L(\Sigma)(M(\Sigma))$ is the set of all $K \in D(X)$ such that $K \leq A$ for some $A \in \Sigma$ ($K \geq A$ for some $A \in \Sigma$). If $0 < t \leq 1$, then $L(t) = L(\mu^{-1}(t))$ and $M(t) = M(\mu^{-1}(t))$.

Theorem 1. If X *is a continuum, then* $H^p(X) \cong H^{p+1}(D(X))$ *for each* $p = 0,1,\ldots$

Proof. Consider the exact sequence:

$$H^p(C(X)) \to H^p(\hat{X}) \to H^{p+1}(C(X),\hat{X}) \to H^{p+1}(C(X)).$$

Since $H^p(C(X)) = 0 = H^{p+1}(C(X))$ by [4], then $H^p(X) \cong H^p(\hat{X}) \cong H^{p+1}(C(X),\hat{X})$. But $D(X) = C(X)/\hat{X}$ and the Map Excision Theorem yields $H^{p+1}(D(X)) \cong H^{p+1}(C(X),\hat{X})$.

Theorem 2. If X *is a continuum and* Σ *is a closed subset of* $D(X)$, *then* $H^1(L(\Sigma)) = 0$.

Proof. The proof is reminiscent of Wallace's Acyclicity Theorem [8]. Suppose $0 \neq e \in H^1(L(\Sigma))$. Use Zorn's Lemma and the Reduction Theorem in cohomology to get a minimal closed Σ such that $e|L(\Sigma) \neq 0$. Since for each $K \in C(X)$, $L(K)$ is homeomorphic to D(K), then $H^1(L(K)) \cong H^1(D(K)) \cong H^0(K) = 0$. Hence Σ is nondegenerate. Let $\Sigma = S \cup T$ for two proper closed sets S and T. Consider the exact sequence:

$$0 = H^0(L(S) \cap L(T)) \to H^1(L(\Sigma)) \to H^1(L(S)) \times H^1(L(T)).$$

Then $e|L(S) = 0$ and $e|L(T) = 0$ contradict the last homeomorphism being one-to-one.

Theorem 3. If X *is a continuum and* $0 < t \leq 1$ *and for each* $K \in \mu^{-1}(t)$, $H^1(K) = 0$, *then* $H^2(L(\Sigma)) = 0$ *for each closed set* Σ *in* $\mu^{-1}(t)$.

Proof. The proof is similar to that of Theorem 2 and uses the fact that $H^1(L(\Sigma)) = 0$ for each closed Σ in D(X).

Theorem 4. Let X *be a continuum. Then*

(a) there is a 1-1 homomorphism $H^1(\mu^{-1}(t)) \to H^1(X)$

(b) if for each $K \in \mu^{-1}(t)$, $H^1(K) = 0$, *then* $H^1(\mu^{-1}(t)) \cong H^1(X)$.

Proof. Consider the exact sequence:

$$\begin{array}{ccccccc} H^1(M(t)) \times H^1(L(t)) & \to & H^1(\mu^{-1}(t)) & \overset{\Delta}{\to} & H^2(D(X)) & \to & H^2(M(t)) \times H^2(L(t)). \\ \| & & & & \| & & \\ 0 & & & & H^1(X) & & \end{array}$$

Then Δ is always 1-1. Since M(t) is acyclic and the hypothesis in (b) and Theorem 3 imply $H^2(L(t)) = 0$, then Δ is onto.

There are many applications of Theorem 4 which Rogers stated in [5]. The next theorem shows that for certain X, those Whitney continua close to the base have the same cohomology.

Theorem 5. If X *is a 1-dimensional continuum and* $H^1(X)$ *is finitely generated over a ring* R *(e.g., cohomology over the integers), then there exists* $0 < t < 1$ *such that* $H^1(X) \cong H^1(\mu^{-1}(s))$ *for each* $s \leq t$.

Proof. Let G be a finite set of generators for $H^1(X)$ as an R-module. For each $g \in G$, $g|\{x\} = 0$. By the Reduction

Theorem, there exists an open set U_x containing x such that $g|U_x = 0$. Let L be a Lebesgue number for $\{U_x | x \in X\}$. Then $g|M = 0$ for each M with diameter < L. Choose L to work for all $g \in G$. Since each element of $H^1(X)$ is a linear combination of elements in G, then $e|M = 0$ for each $e \in H^1(X)$ and diam M < L.

Choose $0 < t < 1$ such that if $\mu(K) \leq t$, then diam $K \leq L$. Let $e \in H^1(K)$ where $\mu(K) \leq t$. Then there exists $f \in H^1(X)$ such that $f|K = e$ since X is 1-dimensional. Then $f|K = 0$ since diam K < L. Hence $H^1(K) = 0$. By Theorem 4, $H^1(\mu^{-1}(s)) \cong H^1(X)$.

References

[1] H. Cohen, *A cohomological definition for locally compact Hausdorff spaces*, Duke Math. J. 21 (1954), 209-224.

[2] J. Krasinkiewicz, *On the hyperspaces of snake-like and circle-like continua*, Fund. Math. LXXXIII 91974), 155-164.

[3] ________ and S. Nadler, *Whitney properties*, to appear Fund. Math.

[4] A. Y. W. Lau, *Acyclicity and dimension of hyperspace of subcontinua*, Bull. Pol. Acad. Sci. XXII, #11 (1974), 1139-1141.

[5] J. T. Rogers, Jr., *Whitney continua in the hyperspace* C(X), Pac. J. Math. 58, #2 (1975), 569-584.

[6] ________, *Applications of a Vietoris-Begle theorem for multi-valued maps to the cohomology of hyperspaces*, to appear Michigan J. Math.

[7] E. Spanier, *Algebraic Topology*, McGraw-Hill Books, 1966.

[8] A. D. Wallace, *A theorem on acyclicity*, Bull. Amer. Math. Soc. 67 (1961), 123-124.

North Texas State University

Denton, Texas 76203