TOPOLOGY PROCEEDINGS
Volume 1, 1976
Pages 191–199
http://topology.auburn.edu/tp/

LUZIN SPACES

by

Kenneth Kunen

Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: 0146-4124

LUZIN SPACES [1]

Kenneth Kunen [2]

0. Introduction

Recall that a subset N of a topological space Y is called *nowhere dense* (n.w.d.) iff the closure of N has empty interior. A *Luzin set* in Y is an uncountable $X \subseteq Y$ such that $X \cap N$ is countable for all n.w.d. $N \subseteq Y$. Luzin [L] showed that the Continuum Hypothesis (CH) implies that there is a Luzin set in the real line, $\mathbb{R}$. More recently, van Douwen, Tall, and Weiss [vDTW] have produced, under CH, Luzin sets in a wide variety of spaces. In this paper we show that, assuming Martin's axiom (MA) plus $\neg$ CH, there are no interesting examples of Luzin sets in any space (see Rudin [R] for an introduction to the use of MA in topology).

We must first dispense with one triviality. Suppose that Y had an uncountable set, X, of isolated points. Then $X \cap N$ is empty for all n.w.d. $N \subseteq Y$, so X is a Luzin set in Y. However,

0.0. *Theorem* (MA + $\neg$ CH). *If* Y *is* T_2 *and has at most countably many isolated points, then there are no Luzin sets in* Y.

The restriction to T_2 spaces is necessary, since any set Y with the cofinite topology is T_1 and a Luzin subset of itself. From now on, all spaces are assumed to be T_2.

In the case $Y = \mathbb{R}$, Theorem 0.0 is an easy consequence of the fact that (under MA + $\neg$ CH), all subsets of $\mathbb{R}$ of cardinality

---

[1] Research supported by N.S.F. Grant GP-43882X.

[2] The author is grateful to the referee for a number of helpful comments.

$\aleph_1$ are of first category, but, as we point out in §3, that fact, or any of the other standard MA consequences about $\mathbb{R}$, is insufficient to imply 0.0 for general Y. 0.0 is proved in §2. §1 collects some preliminary remarks.

1. Background

We review here some known results.

1.0. *Lemma. If* Y *has at most countably many isolated points and* X *is a Luzin set in* Y, *then* X *has no uncountable discrete subsets.*

Proof. A discrete set of non-isolated points is n.w.d.

In other words, X has spread $\aleph_0$, or every subspace of X has the countable chain condition (c.c.c.).

It is useful to rephrase the question of the existence of Luzin sets in terms of intrinsic properties of the set itself. We shall call a *Luzin space* a T_2 space X such that

a) Every n.w.d. set in X is countable (i.e., X is a Luzin set in X).

b) X has at most countably many isolated points.

c) X is uncountable.

If X is a Luzin set in Y and Y has at most countably many isolated points, then X is a Luzin space, since (b) follows by Lemma 1.0 and (a) follows from the fact that every n.w.d. set in X is n.w.d. in Y. Thus, Theorem 0.0 is equivalent to the statement that there are no Luzin spaces. A similar argument also shows

1.1. *Lemma. If* X *is a Luzin space, so is every uncountable subspace.*

Luzin spaces not only have spread $\aleph_0$; they have height $\aleph_0$; i.e., 1.2. *Lemma. If* X *is a Luzin space,* X *is hereditarily Lindelöf* (HL).

Proof. By 1.1, we need only show that X in Lindelöf. Let u be an open cover of X. Since X has c.c.c., there is a countable $v \subseteq u$ with $\bigcup v$ dense in X. Then $X \setminus \bigcup v$ is n.w.d. and hence countable, so there is a countable $v' \subseteq u$ with $(\bigcup v) \cup (\bigcup v') = X$.

Of course, a Luzin subspace of $\mathbb{R}$ is hereditarily separable (HS) as well, but Luzin spaces in general are often not. In fact, van Douwen, Tall, and Weiss [vDTW] show that under CH the method of Luzin spaces is a very useful one for constructing various types of L-spaces (HL spaces which are not HS). 0.0 shows that this method is closed to us under MA + $\neg$ CH. By a different argument (Hajnal-Juhász [HJ]), there is always a T_2 L-space regardless of the axioms of set theory. It is unknown whether MA + $\neg$ CH implies that there are no T_3 L-spaces.

The next lemma shows that, in refuting Luzin spaces, we need only consider ones with no isolated points at all. Call X *everywhere uncountable* iff every non-empty open set in X is uncountable. Then

1.3. *Lemma. If* X *is a Luzin space, there is a* $Z \subseteq X$ *with* $X \setminus Z$ *countable and* Z *everywhere uncountable.*

Proof. Let $W = \bigcup \{V : V \text{ is open in } X \text{ and } |V| \leq \aleph_0\}$. Since X is HL, $|W| \leq \aleph_0$. Let $Z = X \setminus W$.

We recall here the standard construction of a Suslin tree from a Suslin line. Let Y be a Suslin line (a linear c.c.c. connected non-separable space). By passing to a suitable interval, we may assume that Y is everywhere Suslin--i.e., all countable sets are n.w.d. Then define a tree $T = \bigcup \{T_\alpha : \alpha < \omega_1\}$, where T_α is the set of elements on level α. T_α, defined by induction on α, will be a maximal collection of non-empty open intervals. T is ordered by reverse inclusion. $T_0 = \{Y\}$. Given T_α, $T_{\alpha+1}$ is chosen so that every interval in $T_{\alpha+1}$ is properly contained in some interval of T_α. If γ is a limit, T_γ is defined to be the set of non-empty intervals of the form $\mathrm{int}(\bigcap \{J_\xi : \xi < \gamma\})$, where each $J_\xi \in T_\xi$. To show that T_γ is maximal (i.e., $\bigcup T_\gamma$ is dense), we use the fact that the set of endpoints of the intervals in $\bigcup\{T_\xi : \xi < \gamma\}$ is countable and hence n.w.d. Once we have T, the fact that T is Suslin follows immediately from the fact that Y is c.c.c.

A curious sidelight to the construction is that, although in most spaces one needs CH to construct a Luzin set, Suslin lines are an exception:

1.4. *Lemma. If* Y *is a Suslin line, then there is a Luzin set in* Y.

Proof. Let $V_\alpha = \bigcup T_\alpha$. Then the V_α are decreasing and $\bigcap\{V_\alpha : \alpha < \omega_1\} = 0$. In a Suslin line, n.w.d. sets are separable; it follows that for every dense open $W \subseteq Y$, $\exists\alpha(V_\alpha \subseteq W)$. So pick $x_\alpha \in V_\alpha$ and let $X = \{x_\alpha : \alpha < \omega_1\}$.

A Luzin space need not be T_3, since one can always tack on a countable T_2, non-T_3 space to a given Luzin space. However,

1.5. *Lemma. If* X *is a* T_3 *Luzin space, then* X *is* 0-*dimensional (i.e., the clopen sets form a basis).*

Proof. Let $x \in X$ and let U be a neighborhood of x. We find a clopen B with $x \in B \subseteq U$. Since X is T_3 and Lindelöf, X is normal. Let $f : X \to [0,1]$ with $f(x) = 0$ and f equal to 1 outside U. If f is not onto, fix $r \notin \mathrm{range}\ (f)$, and let $B = f^{-1}[0,r) = f^{-1}[0,r]$. However, if f were onto, let K_α $(\alpha < \omega_1)$ be disjoint perfect subsets of [0,1]. Then each $f^{-1}K_\alpha$ is uncountable and closed, so $\mathrm{int}(f^{-1}K_\alpha) \neq 0$, so $\{\mathrm{int}(f^{-1}K_\alpha) : \alpha < \omega_1\}$

would contradict c.c.c.

Remark. For a less trivial example of a non-T_3 Luzin space: assume CH, let $X \subset \mathbb{R}$ be an everywhere uncountable Luzin set, and then refine the topology of X as per [HJ] to make X into a left-separated T_2 L-space. With this topology, X is still an everywhere uncountable Luzin space, and no uncountable subspace of X is T_3.

2. Proof of Theorem 0.0

We shall actually apply MA + $\neg$ CH twice. Our plan is to mimic the construction of a Suslin tree from a Suslin line, and produce a Suslin tree from a Luzin space, contradicting MA + $\neg$ CH (see [R], §5). However, since it is consisted to have CH (and hence Luzin spaces) but no Suslin trees, we cannot expect this construction to work too trivially. We must in a first application of MA + $\neg$ CH, prove enough about Luzin spaces to insure that the tree construction works; this is done in Lemma 2.3, which says that the regular open sets of a Luzin space behave sufficiently like the open intervals of a Suslin line. 2.3 will be an easy consequence of the following fact about the Cantor space, 2^ω:

2.0. *Lemma* (MA + $\neg$ CH). *Suppose* $Z \subseteq 2^\omega$ *and* $|Z| \geq \aleph_1$. *Then there is a family* $\{K_\alpha : \alpha < \omega_1\}$ *of disjoint closed subsets of* 2^ω *such that* $|K_\alpha \cap Z| \geq \aleph_1$ *for each* α.

Proof. Let Z_α $(\alpha < \omega_1)$ be disjoint subsets of Z of cardinality $\aleph_1$. Now, by MA + $\neg$ CH, we know that each Z_α is of first category (see [R], §14), so there are closed n.w.d. $K_{\alpha n}$ $(n < \omega)$ with $Z_\alpha \subseteq \bigcup_n K_{\alpha n}$. We shall show that we can force $K_{\alpha n}$ to be disjoint from $K_{\beta m}$ for all n, m and all $\alpha \neq \beta$. We may then simply let K_α be one of the $K_{\alpha n}$ which covers $\aleph_1$ points of Z_α.

Let $\mathbb{P}$ be the set of pairs $\langle p,a\rangle$ such that

a) p and a are functions with domain $\omega_1 \times \omega$.

b) For all α and n, $p(\alpha,n)$ is a clopen subset of 2^ω, $a(\alpha,n)$ is a finite subset of Z_α, and $a(\alpha,n) \subseteq p(\alpha,n)$.

c) For all but finitely many $\langle \alpha,n\rangle$, $p(\alpha,n) = 2^\omega$ and $a(\alpha,n) = 0$.

Define $\langle p',a'\rangle \le \langle p,a\rangle$ iff $\forall \alpha n[p'(\alpha,n) \subseteq p(\alpha,n)$ and $a'(\alpha,n) \supseteq a(\alpha,n)]$.

(c) insures that the conditions $\langle p,a\rangle$ are essentially finitary. The standard Δ-system argument shows that $\mathbb{P}$ has the c.c.c. (See [R], §§16, 17 for other such arguments).

Intuitively, $\langle p,a\rangle$ says, "For all α and n, $a(\alpha,n) \subseteq K_{\alpha n} \subseteq p(\alpha,n)$"; the presence of the a part of our conditions enables us to force Z_α to be a subset of $\bigcup_n K_{\alpha n}$. More formally, if G is a filter in $\mathbb{P}$, define $K_{\alpha n} = \bigcap\{p(\alpha,n) : \exists a(\langle p,a\rangle \in G)\}$. So $K_{\alpha n}$ is closed. To make $Z_\alpha \subseteq \bigcup_n K_{\alpha n}$ insure that G intersects the dense set $\{\langle p,a\rangle : \exists n[z \in a(\alpha,n)]\}$ for each $z \in Z_\alpha$. To make $K_{\alpha n} \cap K_{\beta m} = 0$, make sure that G intersects $\{\langle p,a\rangle : p(\alpha,n) \cap p(\beta,m) = 0\}$, which is dense whenever $\alpha \neq \beta$ since $Z_\alpha \cap Z_\beta = 0$. This proves 2.0. If we want also $K_{\alpha n}$ to be n.w.d., make G intersect the sets $\{\langle p,a\rangle : F \not\subseteq p(\alpha,n)\}$ for each clopen F and each α,n.

2.1. *Lemma* (MA + $\neg$ CH). *If* X *is a Luzin space and* $f : X \to 2^\omega$ *is continuous, then the range of* f *is countable.*

Proof. Let Z = ran f. If $|Z| \ge \aleph_1$, let K_α be as in 2.0. Then as in the proof of 1.5, $\{\text{int}(f^{-1}K_\alpha) : \alpha < \omega_1\}$ would contradict c.c.c.

2.2. *Lemma* (MA + $\neg$ CH). *Let* X *be a Luzin space and* F_n $(n \in \omega)$ *clopen sets in* X. *Define* $x \sim y$ *iff* $\forall n[x \in F_n \iff y \in F_n]$. *Then there are only countably many equivalence classes under* $\sim$.

Proof. Let χ_{F_n} be the characteristic function of F_n. Define $f : X \to 2^{\omega}$ by $(f(x))(n) = \chi_{F_n}(x)$, and apply 2.1.

2.3. *Lemma* (MA + $\neg$ CH). *Let* X *be an everywhere uncountable Luzin space. Let* U_k^n $(n,k \in \omega)$ *be regular open sets in* X *such that for each* n,

a) *The* U_k^n $(k \in \omega)$ *are disjoint and*

b) $\bigcup_k U_k^n$ *is dense.*

Then

$$\bigcup\{int(\bigcap_n U_{h(n)}^n) : h \in \omega^{\omega}\}$$

is dense in X *(*ω^{ω} *is the set of functions from* ω *to* ω*).*

Proof. Suppose not. For $h \in \omega^{\omega}$, let $E_h = \bigcap_n U_{h(n)}^n$. Let W be a non-empty open subset of X disjoint from each $int(E_h)$. Let

$$A = \left(\bigcup_{n,k} cl(U_k^n)\setminus U_k^n\right) \cup \left(\bigcup_n (X\setminus \bigcup_k U_k^n)\right).$$

Since X is a Luzin space, A is countable, and since X is everywhere uncountable W\A is uncountable and hence a Luzin space by 1.1. For each n, $\bigcup_k(U_k^n \cap (W\setminus A)) = W\setminus A$, and each $U_k^n \cap (W\setminus A)$ is clopen in W\A. Applying 2.2 to W\A and the $U_k^n \cap (W\setminus A)$, $E_h \cap (W\setminus A) = 0$ for all but countably many h, so some $E_h \cap (W\setminus A)$ is uncountable, so $E_h \cap W$ is uncountable. Let V be a non-empty open subset of W with $V \subseteq cl\ E_h$. Then $cl(V) \subseteq cl(E_h) \subseteq \bigcap_n cl\ U_{h(n)}^n$. Since $U_{h(n)}^n$ is regular open, $int\ cl(V) \subseteq \bigcap_n U_{h(n)}^n = E_h$, contradicting that $W \cap int(E_h) = 0$.

We now prove 0.0. Suppose there is a Luzin space, X. By 1.3, we may assume X is everywhere uncountable. Define T verbatim as in the paragraph following 1.3, except that we replace "open interval" by "regular open set." When γ is a limit, we use 2.3 (which holds by MA + $\neg$ CH) to check that $\bigcup T_{\gamma}$ is dense; i.e., we let $\gamma = \{\alpha_n : n \in \omega\}$, and, for each n, let U_k^n $(k \in \omega)$ be a 1 - 1 enumeration of T_{α_n}, and apply 2.3 to these U_k^n

(technical point: if T_{α_n} is finite, let U_k^n $(k < j)$ enumerate T_{α_n} and $U_k^n = 0$ for $k \geq j$). So, the construction works to give a Suslin tree, which, by MA + $\neg$ CH, is a contradiction.

Perhaps a slicker proof would be to simply remark that 2.3 says that the regular open algebra is a Suslin algebra (i.e., a complete, non-atomic, c.c.c., (ω,ω)-distributive Boolean algebra), since this is known to yield a Suslin tree (by the same construction as in the above paragraph).

3. Conclusion

We may examine more closely our double use of MA + $\neg$ CH. The proofs above show that

MA + $\neg$ CH => Lemma 2.0, and

Lemma 2.0 => (There exists a Luzin space <=>
There exists a Suslin tree).

One then quotes the known result that under MA + $\neg$ CH, there are no Suslin trees.

2.0 itself is quite strong, eliminating most examples of Luzin spaces. 2.0 implies (via 2.3) that no Luzin space can have a dense subspace with a coarser second countable T_1 topology. In particular, under 2.0, there are no separable Luzin spaces, and there are no Sierpiński sets of reals (which would be Luzin sets in the density topology--see Tall [T]).

However, 2.0 alone does not imply that there are no Luzin spaces at all. To see this, recall that a partial order $\mathbb{P}$ is said to have property K (for Knaster) iff every uncountable subset of $\mathbb{P}$ has an uncountable subset which is pairwise compatible. An example of such is the $\mathbb{P}$ used in proving 2.0 (in fact, this $\mathbb{P}$ has ω_1 as a precaliber). Let MA_K be Martin's axiom restricted to partial orders with property K; so MA_K + $\neg$ CH => 2.0. Since it is consistent with MA_K + $\neg$ CH to have a Suslin tree (see [KT]), it is consistent with 2.0 to have a Luzin space.

Finally, it is pointed out in [KT] that $MA_K + \neg CH$ implies all the standard MA consequences regarding measure and category in $\mathbb{R}$. Thus, as we remarked at the end of §0, these facts alone are insufficient to imply Theorem 0.0.

References

[vDTW] E. K. van Douwen, F. D. Tall, and W. A. R. Weiss, *Nonmetrizable hereditarily Lindelöf spaces with point-countable bases from* CH, to appear.

[HJ] A. Hajnal and I. Juhász, *On hereditarily α-Lindelöf and α-separable spaces*, Ann. Univ. Sci. Budapest 11 (1968), 115-124.

[KT] K. Kunen and F. D. Tall, *Between Martin's axiom and Souslin's hypothesis*, Fund. Math., to appear.

[L] N. Lusin, *Sur un problème de M. Baire*, Comptes Rendus (Paris) 158 (1914), 1258-1261.

[R] M. E. Rudin, *Martin's axiom*, to appear in Handbook of Mathematical Logic, North Holland, 1977.

[T] F. D. Tall, *The density topology*, Pacific J. Math., 62 (1976) 275-284.

University of Wisconsin

Madison, Wisconsin 53706