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1. A *continuum* is a compact connected Hausdorff space. A closed subset A of a continuum X is a *nodal set* if Fr (A) contains at most one point. A subcontinuum C of X is a *universal subcontinuum* (USC) of X if $C \cap A$ is a continuum for each subcontinuum A of X. A nodal set is a USC. The intersection of arbitrarily many USC's is a USC. X is *semi-locally connected* (s.l.c.) if each point of X has arbitrarily small neighborhoods whose complements have a finite number of components. A locally connected continuum is s.l.c. A point $x \in X$ *separates* y from z if $X - x = U|V$ with $y \in U$ and $z \in V$. We write $E(x,y) = \{x,y\} \cup \{z: \text{ z separates x from y}\}$. A point e is an *endpoint* of X if e has arbitrarily small neighborhoods whose closures are nodal sets.

1.1. LEMMA: Let X *be a continuum.*

(a) Let α *be a collection of USC of* X. α *has the finite intersection property if any two elements of* α *meet. Then if* C *is a subcontinuum of* X, $\alpha \cup \{C\}$ *has the finite intersection property iff* C *meets each member of* α.

(b) If A *and* B *are USC of* X *which meet, then* $A \cup B$ *is a USC.*

The proof of (a) is obtained by modifying the argument of [3], Lemma 1. (b) is proved in [6], where a USC is called a semi-chain.

1.2. THEOREM: Let X *be a s.l.c. continuum, and* x, y, *and* $z \in X$.

(a) If $x \notin E(y,z)$*, y and z lie in a continuum* $C \subset X-x$.

(b) Each component of $X-x$ *is open.*

(c) If x *is a non-cutpoint,* x *has arbitrarily small neighborhoods whose complements are connected.*

(d) Each USC of X *is s.l.c.*

(e) If C *is a USC of* X *and* $x, y, z \in C$ *are such that* z *separates* x *from* y *in* C*, then* z *separates* x *from* y *in* X.

The proof is left to the reader.

By 1.2(a), the set $E(x,y)$, is closed; order $E(x,y)$ as follows: x is the least and y is the largest element. If $z, w \in E(x,y)$, then $z \leq w$ if $z \in E(x,w)$. $\leq$ is a total order, called the *separation order* on $E(x,y)$. The order topology on $E(x,y)$ is the same as the subspace topology.

2. Cyclic element theory for metric spaces is found in [8]. We will consider non-metrizable continua, so we will develop certain results in cyclic element theory which will be needed later. Certainly not all the results of this section are new, but the proofs seem to be new.

In this section X is a s.l.c. continuum. A *true cyclic element* of X is a subcontinuum C of X which is maximal with respect to containing no cutpoint of itself. Any connected subset of X which contains no cutpoint of itself is contained in a true cyclic element. Two points $x, y \in X$ are conjugate, $x \sim y$, if $E(x,y) = x \cup y$. We write: $C(x) = \{y \in X: \ x \sim y\}$

Any two points of a true cyclic element are conjugate.

2.1. THEOREM: *Let* X *be a s.l.c. continuum and* $x \in X$.

(a) $C(x)$ *is the intersection of all nodal sets* A *such that* $x \in$ *Int* A.

(b) If x *is a non-cutpoint for which* $C(x)$ *is degenerate, then* x *is an endpoint.*

(c) If C(x) *is non-degenerate,* C(x) *is the union of all true cyclic elements of* X *which contain* x*;* C(x) *can have no cutpoint except, possibly,* x*; if* R *is a component of* C(x)-x*, then* $\overline{R}$ *is a true cyclic element.*

(d) If two distinct cyclic elements meet, they meet in a cutpoint of X.

The proof is left to the reader.

A *cyclic element* of a s.l.c. continuum X is an endpoint, cutpoint, or a true cyclic element. If C is a true cyclic element of X, and x, y are distinct points of C, then $C = C(x) \cap C(y)$, so a point $z \notin C$ can be conjugate to at most one point of C. If X is metrizable, each true cyclic element contains a non-cutpoint of X [8]. The following example shows that this is not so in general.

2.2. *EXAMPLE:* Let S' denote the unit circle. For each $x \in S'$, let $I_x = [0,1]$ and $Y = \Pi\{I_x: x \in S'\}$. Let $P_x: Y \to I_x$ be the x-th projection. Let: $X = \{(x,f)\ S' \times Y: f(y) = 0$ for $y \in S' - x\}$. X is closed in the product $S' \times Y$, hence is compact. Certainly X is connected. Each point $(x,f) \in X$ for which $f(x) < 1$ is a cutpoint. Finally X is locally connected.

The only true cyclic element of X is $S' \times 0$. This space has only endpoints and cutpoints, but is not a tree. Wallace [6] points out that such spaces exist, but does not give an example.

2.3. *THEOREM: If* C *is a true cyclic element of a s.l.c. continuum* X *and* $x \notin C$*, some point of* C *separates a point of* C *from* x.

Proof: Suppose the theorem is false. Choose $y \in C$ for which $y \not\subset C(x)$. Then E(y,x) contains a point w for which $y < w < x$, where $\leq$ is the separation order on E(y,x). Choose $z \in C-y$. Then y cannot separate z from x, so $z \cup x$ is contained in a subcontinuum $D \subset X-y$. Then $E(x,y)-y \subset D$, hence $y \notin \overline{E(x,y)-y}$. Since $E(x,y)$ is a compact ordered space, there is a first element $w \in E(y,x)$ following y.

Let R be the component of $X-C$ containing x. Since $\overline{R} \cup C$ is a continuum we have $E(y,x) \subset \overline{R} \cup C$, and this means $w \in R$.

Now if $y \sim w$, we cannot have $z \sim w$, for otherwise $w \in C$. But since $z \sim y$ and $w \sim y$, we must have $y \in E(z,w)$, i.e. $X-y - U|V$, where U is the component of $X-y$ containing z, and $w \in V$. Then $R \subset V$ and $y \in E(a,x)$ which is impossible. If $y \nsim w$, then $E(y,w) - (y \cup w) \neq \emptyset$, and this contradicts the choice of w. The proof is complete.

2.4. THEOREM: Let C *be a true cyclic element of a s.l.c. continuum* X. *Then the components of* $X-C$ *are open sets with one point boundaries.*

Proof: Let R be a component of $X-C$ and choose $x \in R$. By 2.3, we can find $y \in C$ and a separation $X-y = U|V$ where U is a component of $X-y$ with $C \cap U \neq \emptyset$ and $x \in V$. Then $C \subset U \cup y$ and $R \subset V$. Thus R is a component of V, hence of $X-y$.

Let X be a s.l.c. continuum and $A \subset X$. The convex hull of A, $H(A)$, is the union of A and all points which separate a pair of points of A. $H(A)$ contains A and contains $E(x,y)$ whenever $x,y \in H(A)$, and is the smallest subset of X with these properties.

2.5. LEMMA: Let X *be a s.l.c. continuum and* A *be a closed subset of* X. *If* x *is not in the convex hull of* A, A *lies in a subcontinuum of* X *not containing* x.

Proof: Choose a neighborhood U of x which does not meet A, and whose complement has finitely many components. Let $C_1, \cdots, C_n$ be those components of $X-U$ which meet A. Since x separates no pair of points of A, by 1.2(a), we can link C_i and C_{i+1} with a continuum K_i lying in $X-x$, $i=1, \cdots, n-1$. The desired continuum is $C_1 \cup K_1 \cup C_2 \cup \cdots \cup K_{n-1} \cup C_n$.

2.6. COROLLARY: The convex hull of a closed subset of a semi-locally connected continuum is closed.

2.7. THEOREM: Let F *be a closed non-degenerate subset of a s.l.c. continuum* X. *Suppose* $x \notin H(F)$ *and* x *is conjugate to at most one point of* H(F). *Then there is a nodal set* A *with* $F \subset A$ *and* $x \notin A$.

Proof: Suppose x lies in a true cyclic element C of X. Then C contains at most one point of F, so F meets a component R of X-C. Let $y = \overline{R} \cap C$. Let S be the union of all components of X-y which lie in X-C and meet F. If F meets a component R_1 of X-C with $\overline{R}_1 \cap C = z \neq y$, then $x \sim y$, $x \sim z$ and $y \cup z \subset H(F)$, and this is impossible. Suppose $y \neq x$. Then C contains no member of F other than y, for otherwise y would be in H(F). Thus $S \cup y$ is the required nodal set. If y = x, then S is a component of X-x (because $x \notin H(F)$) and $F \subset S$. If x lies in a true cyclic element C_1 of $\overline{S}$, we repeat the above argument with $\overline{S}$ replacing X and find a point y_1 and the union S_1 of the components of $\overline{S}-y_1$ which meet F and lie in $\overline{S}-C_1$. This time $x \neq y_1$ since S is connected, and $S_1 \cup y_1$ is the required nodal set. If x lies in no true cyclic element of $\overline{S}$, replace X by $\overline{S}$ in the following argument.

Suppose C(x) = x. By 2.5, F lies in a continuum $C \subset X-x$. Using 1.1(a) and 2.1(a), we find that C fails to meet a nodal set A with $x \in \text{Int } A$. Then $\overline{X-A}$ is the required nodal set.

If M is a closed subset of X let A(M) be the intersection of all nodal sets containing M. M is an *A-set* if A(M) = M.

2.8. THEOREM: If M *is a closed subset of a s.l.c. continuum* X, *then* A(M) *is the union of* H(M) *and all true cyclic elements of* X *which meet* H(M) *in at least two points.*

Proof: If B is a nodal set containing M then B contains H(M) and any true cyclic element of X meeting H(M) in at least two points.

If $x \notin H(M)$ and x is not in any true cyclic element meeting H(M) in two points, then x can be conjugate to at most one point of H(M): if $x \sim y$ and $x \sim z$ with $y, z \in H(M)$ and $y \neq z$, then $x \cup y$ and $x \cup z$ lie in true cyclic elements E and F of X. Now $E = F$ is impossible and $E \neq F$ implies $x = E \cap F$ so $x \in H(M)$, which also is impossible. Hence there is a nodal set B with $M \subset B$ and $x \notin B$. This means $x \notin A(M)$ and the proof is complete.

2.9. THEOREM: *If* M *is a closed nonempty subset of a s.l.c. continuum* X, *then* A(M) *is the union of cyclic elements of* X.

Proof: We show that if $x \in A(M)$, then there is a cyclic element C of x with $x \in C \subset A(M)$. If x is an endpoint or a cutpoint, this is obvious. Otherwise, x is a non-cutpoint other than an endpoint. If B is a nodal set with $M \subset B$, then $x \in \text{Int } B$, so by 2.1(a), $C(x) \subset B$. Thus $C(x) \subset A(M)$, and by 2.1(c), C(x) is a true cyclic element.

3. Our terminology for transformation groups comes from [2], except our groups act on the left.

Let (X,T) be a transformation group, where X is a continuum. X is *T-irreducible* if no proper subcontinuum of X is invariant. If X is T-irreducible, then no proper USC of X can contain a nonempty invariant set I. For otherwise, the intersection of all USC of X containing I would be invariant.

3.1. THEOREM: *Let* (X,T) *be a transformation group, where* X *is a s.l.c. continuum which is* T-*irreducible and contains a cutpoint.*

(a) X *contains at least two endpoints.*

(b) If e *is an endpoint of* X, *then* $e \in \overline{Tx}$ *for each* $x \in X$.

(c) (X,T) *is almost periodic at each endpoint of* X, *and* X *contains exactly one minimal orbit closure.*

(d) If (X,T) *is pointwise almost periodic, then* X *is a minimal orbit closure.*

Proof: To prove (a), we will show that if z is any non-cutpoint of X, then X contains an endpoint different from z.

We first observe that if $X-x = U|V$, then neither $U \cup x$ nor $V \cup x$ can contain the orbit Tx, so both contain a cutpoint of X (see the remarks preceding 3.1).

Let E be the set of cutpoints of X. For each $x \in E$ write $X-x = R(x)|S(x)$ where $R(x)$ is the component of $X-x$ containing z. Now $y \in S(x) \cap E$ implies $\overline{S(y)} \subset S(x)$: For $\overline{R(x)}$ is connected, meets $R(y)$, and $y \notin R(x)$, so $\overline{R(x)} \subset R(y)$. A similar argument then shows that $\overline{S(y)} \subset S(x)$.

Order the collection $\{\overline{S(x)}: x \in E\}$ by inclusion and extract a subcollection $\{\overline{S(y)}: y \in Y\}$ which is maximal with respect to being totally ordered by inclusion. Let $S = \bigcap \{\overline{S(y)}: y \in Y\}$. We prove that S contains no cutpoint of X. For if $x \in S$ is a cutpoint of X, then $x \in S(y)$ for each $y \in Y-x$ so $\overline{S(x)} \subset S(y)$ for such y. Now $S(x)$ contains a cutpoint p of X and $\overline{S(p)} \subset S(x)$. Then $\{\overline{S(y)}: y \in Y\}$ is not maximal, which is a contradiction.

Since S is a USC and contains no cutpoints of X, 1.2(e) implies that S has no cutpoints of itself, and by contruction, S is maximal with respect to this property. By 2.3, each true cyclic element contains a cutpoint of X, so S is an endpoint. This proves (a).

(b) Suppose e is an endpoint and $e \notin \overline{Tx}$ for some $x \in X$. Then there is a nodal set A with $e \in \operatorname{Int} A$ and $A \subset X-\overline{Tx}$, whence $\overline{Tx} \subset \overline{X-A}$. Thus X is not T-irreducible.

(c) By [2, 4.06] X contains an almost periodic point, and (X,T) is almost periodic at each point of $\overline{Tx}$ [2, 4.09]. Thus the first reading of (c) follows from (b).

Distinct minimal orbit closures are disjoint; thus the second reading of (c) follows from (b), and [2, 2.12].

(d) If (X,T) is pointwise almost periodic, the orbit closure of each point is minimal. Now use (c).

We now extend some results of Ayres, [1], on pointwise almost periodic homeomorphisms to transformation groups. The proof of the following theorem is to be found in [4] and [7].

3.2. THEOREM: Let (X,T) *be a transformation group, where* X *is a continuum. Suppose one of the following holds:*

(a) T *is abelian.*

(b) (X,T) *is pointwise regularly almost periodic and* X *is s.l.c. Then* T *leaves invariant a subcontinuum of* X *which contains no cutpoints of itself.*

In [4], locally connectedness was assumed for (b), but semilocal connectedness is actually sufficient.

3.3. LEMMA: Let (X,T) *be a pointwise almost periodic transformation group where* X *is a s.l.c. continuum for which* $X = A \cup B$ *where* A *and* B *are nodal sets with* $A \cap B = \{x\}$. *If* tA *meets* A *and* tB *meets* B *for each* $t \in T$, *then* x *is fixed under* T.

Proof: By 1.1(a), the collections $\{tA\colon t \in T\}$ and $\{tB\colon t \in T\}$ have the finite intersection property. Let $C = \bigcap \{tA\colon t \in T\}$ and $D = \bigcap \{tB\colon t \in T\}$. Then C and D are nonempty T-invariant subcontinua of X.

Assume x is not fixed. Choose $t \in T$ for which $tx \neq x$. Suppose $tx \in B$. Since A meets tA and does not contain tx, $A \subset tA$. Then tB meets B and does not contain x, so $tB \subset B$. Thus $x \notin D$. Since $tx \notin A$, we have $tx \notin C$, hence $x \notin C$. Thus x separates each point of C from each point of D. Let

$$H = \bigcap \{E(y,z)\colon y \in C,\ z \in D\}.$$

Then H is T-invariant, and any E(y,z) induces a total order on H under which T acts as a group of order isomorphisms. Since $x \in H$, we have $\overline{Tx} \subset H$, and since $\overline{Tx}$ is a compact ordered space, it contains a largest element which must be fixed under T. But $\overline{Tx}$ is minimal, [2, 4.07], hence contains no fixed point. This contradiction proves the theorem.

3.4. THEOREM: Let (X,T) *be an almost periodic transformation group where* X *is a non-trivial s.l.c. continuum. If* T *leaves an endpoint* e *of* X *fixed, then* T *leaves infinitely many cutpoints of* X *fixed.*

Proof: Choose a neighborhood U of any $x \neq e$ with $e \in U$. Let A be a syndetic subset of T for which $Ax \subset U$, and K be a compact subset of T for which KA = T. Then $\overline{Ax} \subset \overline{U}$, and since e is fixed under T, $e \notin K\overline{Ax} = \overline{Tx}$. Choose a nodal set B with $e \in \text{Int } B$ and $B \subset X - \overline{Tx}$. If y = Fr(B), by 3.3, y is fixed. Since e is an endpoint, E(e,y) is infinite and 3.3 implies that each point of E(e,y) is fixed under T.

3.5. THEOREM: Let (X,T) *be an almost periodic transformation group where* X *is a s.l.c. continuum. Let* I(T) *be the union of all invariant cyclic elements of* X. *If either* T *is abelian or* (X,T) *is pointwise regularly almost periodic, then* I(T) *is a nonempty* A*-set.*

Proof: That $I(T) \neq \emptyset$ follows from 3.2. If I(T) is a point, it is an endpoint or cutpoint, and we are through. Thus assume I(T) is non-degenerate. We first show that if $x \in X$ separates a pair of points of $\overline{I(T)}$, then x is fixed under T. Suppose $X - x = U|V$ where U and V meet $\overline{I(T)}$. Then U and V are open, hence both meet I(T) and $U \cup x$ and $V \cup x$ both contain an invariant cyclic element, so 3.3 implies that x is fixed.

Next we will show that if $x \in \overline{I(T)}$, then x lies in an invariant cyclic element. This will prove that $\overline{I(T)} = I(T)$.

Suppose $x \sim y$ with $y \in I(T)$. Let C be a true cyclic element containing $x \cup y$. We assume that x lies in no invariant cyclic element of X. If $I(T) \subset C$, then the non-triviality of $I(T)$ implies that C is invariant, which is contrary to assumption. Thus $I(T)$ meets a component R of X-C. Let $\overline{R} \cap C = z$. Then z separates two points of $I(T)$, and as we have seen, z must be fixed. Let S be the union of all components of $X - c$ whose closures meet C in the point z and write $X - z = S|U$. Then $x \in U$ so $I(T)$ must meet U. Hence $\overline{U}$ contains an invariant cyclic element A. Now $A \subset C$ would imply that $tC \cap C$ contains two points, all $t \in T$, and so $tC = C$, all $t \in T$, and C would be invariant. Thus A meets a component Q of $X - C$, and $\overline{Q}$ contains an invariant cyclic element. If $W = \overline{Q} \cap C$, then W must be fixed. Then C contains two fixed points, hence is invariant. This contradiction shows that x must lie in some invariant cyclic element of X.

Now suppose $x \not\sim y$ for all $y \in I(T)$. Choose any $y \in I(T)$. Then $E(x,y)$ contains a point $w \notin x \cup y$. Arguing as above, any such point is seen to be fixed. If $E(x,y)$ contains a first element w following x in its separation order then $x \sim w$ and this is not possible since $w \in I(T)$. Since the order topology and subspace topology on $E(x,y)$ coincide, x is a limit point of fixed points, hence is fixed. If x is an endpoint or cutpoint we are through. Otherwise $C(x)$ is a true cyclic element, hence is invariant. Thus $I(T)$ is closed.

We have already seen that each cyclic element in $H(I(T))$ is closed. Now let C be a true cyclic element of X which meets $H(I(T))$ in at least two points, say x and y. Then x and y lie in invariant cyclic elements A and B, respectively. If $A = B$, then $A = B = C$ and C is invariant. Otherwise $A \cap B = \emptyset$: for A, B, and C are USC of X, and $A \cap B = \emptyset$ would imply $A \cap B \cap C = \emptyset$ by 1.1(a), so $A \cap C = B \cap C = A \cap B \cap C$ is a point.

If A is non-degenerate, then $A \cap C$ separates $A - A \cap C$ from B, so is fixed. In any case, $A \cap C$ is a fixed point. Likewise, $B \cap C$ is a fixed point. Then C contains two fixed points, so is invariant.

We have shown that each cyclic element of X lying in $A(I(T))$ is invariant. By 2.9, this means that $A(I(T)) = I(T)$, and the proof is complete.

Note: In the above proof, the fact that I(T) is closed can be deduced by use of Whyburn's H-sets. This direct proof seems simpler, however.

3.6. THEOREM: Let (X,T) *be a pointwise almost periodic transformation group, where* X *is s.l.c. and* T *is either abelian or* (X,T) *is pointwise regularly almost periodic. If* C *is a non-invariant cyclic element,* $A(\overline{TC})$ *contains exactly one invariant cyclic element of* X.

Proof: $A(\overline{TC})$ is an invariant non-trivial subcontinuum of X, hence by 3.2 some cyclic element D of $A(\overline{TC})$ is invariant. If D is an endpoint of $A(\overline{TC})$, 3.4 implies some cutpoint x of $A(\overline{TC})$ is invariant and x is a cutpoint of X. If D is a cutpoint of $A(\overline{TC})$, then D is also a cutpoint of X. If D is a true cyclic element of $A(\overline{TC})$, then D is also a true cyclic element of X (use 2.9). Thus $A(\overline{TC})$ contains an invariant cyclic element of X. If $A(\overline{TC})$ contains two invariant cyclic elements, then 3.3 implies that some cutpoint x separating two points of $\overline{TC}$ is fixed. Let $\{R_\alpha\}$ be the collection of all components of X-x which meet $\overline{TC}$. Since each R_α is open, each $\overline{R}_\alpha$ meets TC, hence each contains tC for some $t \in T$ (t depends on α). If $R_\alpha \neq R_\beta$, let $tC = R_\alpha$ and $sC = R_\beta$. Then $st^{-1}R_\alpha = R_\beta$. Since $A(\overline{TC}) \subset \cup \{R_\alpha\} \cup x$, $A(\overline{TC})$ contains no invariant cyclic element other than x.

As a final note we observe that there are dendrites which are minimal orbit closures under their total homeomorphism groups. One such example was constructed by Doyle and Hocking (Amer. Math. Monthly, 1961). Thus pointwise almost periodicity of a transformation group (X,T) with s.l.c. phase space is not sufficient to insure the existence of an invariant cyclic element.

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