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Assuming MA + ¬ CH, G. M. Reed constructed a separable countably paracompact Moore space that fails to be normal, [3]; some set theoretic assumption beyond ZFC is necessary since W. G. Fleissner showed that CH implies that a separable countably paracompact Moore space must be metrizable, hence normal, [2]. (It is an open question if there exists a countably paracompact nonnormal Moore space in ZFC, this question was raised in [4], and implicitly in [7]. However, it is known that if there exists a nonmetrizable normal Moore space, then there exists a countably paracompact nonnormal Moore space. [5].) Subsequently M. L. Wage constructed a first countable separable countably paracompact CWH (= collectionwise Hausdorff) zero-dimensional space with ω_1 points that fails to be normal, without using any axioms beyond ZFC, [6]. The purpose of this note is to show that the existence of such a space, which in addition is locally compact, follows in a natural way from the existence of a Hausdorff Gap, [2] (see the Lemma below). It seems to me that this construction, including the construction of a Hausdorff Gap, is easier than the construction in [6]. The fact that our construction is related in such a natural way to a classical combinatorial result makes it of independent interest.

Example. There exists a first countable separable countably paracompact CWH locally compact zero dimensional space that fails to be normal.

To explain what a Hausdorff Gap is, we need the following concept. If A and B are sets, then A is almost contained in

$B, A \subseteq^* B$, if $A - B$ is finite; we write $A \subset^* B$ if $A \subseteq^* B$ but not $B \subseteq^* A$.

Lemma. (Hausdorff, [2]) *There are families* $\{A(i,\alpha): \alpha \in \omega_1, i = 0,1\}$ *of subsets of* ω, *the finite ordinals, such that*

(1) $\alpha < \beta$ *implies* $A(i,\alpha) \subset^* A(i,\beta)$, $i = 0,1$,

(2) $A(0,\alpha) \cap A(1,\beta)$ *is finite for all* $\alpha, \beta \in \omega_1$, *and*

(3) There is no $S \subset \omega$ *such that* $A(0,\alpha) \subset^* S$ *and* $A(1,\alpha) \subset^* \omega - S$ *for all* $\alpha \in \omega_1$.

(Actually Hausdorff considered $\{A(0,\alpha): \alpha \in \omega_1\}$ and $\{\omega - A(1,\alpha): \alpha \in \omega_1\}$ which explains the name Hausdorff Gap a little bit better.)

It is quite remarkable that this does not require the Continuum Hypothesis.

Construction of the Example

Let the $A(i,\alpha)$'s be as in the Lemma. For $i = 0,1$ let $L_i = \{i\} \times (\omega_1 - \{0\})$. Topologize $H = L_0 \cup L_1 \cup \omega$ as follows. Points of ω are isolated, and a basic neighborhood of a point $(i,\beta) \in L_0 \cup L_1$ has the form

$B(i,\beta,\alpha,F) = \{(i,\xi): \alpha < \xi \leq \beta\} \cup ((A(i,\beta) - A(i,\alpha)) - F)$

with $\alpha < \beta$ and $F \subset \omega$ finite. It is easily verified that this definition is correct, and that H is a first countable Hausdorff space. Since $A(i,\beta) - A(i,\alpha)$ is infinite whenever $\alpha < \beta$, the set ω is dense, hence H is separable.

In order to show that H is locally compact and zero-dimensional we show that each $B(i,\beta,\alpha,F)$ is compact. (Zero-dimensionality can be proved directly, of course.) Let $\mathcal{U}$ be an open cover of some $B(i,\beta,\alpha,F)$. For some $n \geq 1$ there are $\beta_0, \cdots, \beta_n$, with $\beta = \beta_0 > \cdots > \beta_n = \alpha$, and a finite $G \subset \omega$, such that $\mathcal{B} = \{B(i,\beta_k,\beta_{k+1},G): 0 \leq k < n\}$ covers $\{(i,\xi): \alpha < \xi \leq \beta\}$ and refines $\mathcal{U}$. But then

$$B(i,\beta,\alpha,F) - \bigcup \mathcal{B} \subseteq G \cup [(A(i,\beta_0) - A(i,\beta_n)) - \bigcup_{0 \leq k < n}(A(i,\beta_k) - A(i,\beta_{k+1}))] = G$$

since $(A_0 - A_n) - \bigcup_{0 \leq k < n}(A_k - A_{k+1}) = \emptyset$ for any sets $A_0, \cdots, A_n$, so $B(i,\beta,\alpha,F)$ is compact since G and $\mathcal{B}$ are finite.

The subspaces L_0 and L_1 of H are copies of $\omega_1 - \{0\}$, hence are countably compact. It follows that every countable open cover of H contains a finite subcollection which covers the non-isolated points, hence H is countably paracompact, and it also follows that each closed discrete subset of H consisting of non-isolated points is finite, hence H is CWH.

It remains to show that H is not normal. Assume the contrary. Then the disjoint closed sets L_0 and L_1 would have disjoint neighborhoods, U_0 and U_1 say. For each $\beta \in \beta_1 - \{0\}$ we can choose $f(\beta) < \beta$ and a finite $E(\beta) \subset \omega$, not depending on i (this only is a trick), such that $B(i,\beta,f(\beta),E(\beta)) \subseteq U_i$ for $i = 0,1$. Since $f(\beta) < \beta$ for all $\beta \in \omega_1 - \{0\}$ there is, as well known, a $\kappa \in \omega_1$ and a cofinal $K \subset \omega_1$ such that $f(\beta) = \kappa$ for $\beta \in K$. Define S_0 and S_1 by

$$S_i = [A(i,\kappa) \cup (\omega \cap U_i)] - A(1 - i,\kappa).$$

Then $S_0 \cap S_1 = \phi$ since $U_0 \cap U_1 = \phi$. Let $i = 0,1$ and $\alpha \in \omega_1$ be arbitrary. Since K is cofinal, there is a $\beta \in K$ with $\beta \geq \alpha$, hence with $A(i,\alpha) \subseteq * A(i,\beta)$. But $f(\beta) = \kappa$, hence $(A(i,\beta) - A(i,\kappa)) - E(\beta) \subseteq \omega \cap U_i$, so $A(i,\beta) \subseteq * A(i,\kappa) \cup (\omega \cap U_i)$. Since $A(i,\alpha) \cap A(1-i,\kappa)$ is finite, it follows that $A(i,\alpha) \subseteq * S_i$. Since $i = 0,1$ and $\alpha \in \omega_1$ are arbitrary, and $S_0 \cap S_i = \phi$ this contradicts (3) of the Lemma.

This completes the verification of the properties of H.

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