TOPOLOGY PROCEEDINGS

Volume 1, 1976 Pages 239–242



http://topology.auburn.edu/tp/

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Topology Proceedings

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ISSN: 0146-4124

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HAUSDORFF GAPS AND A NICE COUNTABLY PARACOMPACT NONNORMAL SPACE

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Assuming MA + \neg CH, G. M. Reed constructed a separable countably paracompact Moore space that fails to be normal, [3]; some set theoretic assumption beyond ZFC is necessary since W. G. Fleissner showed that CH implies that a separable countably paracompact Moore space must be metrizable, hence normal, [2]. (It is an open question if there exists a countably paracompact nonnormal Moore space in ZFC, this question was raised in [4], and implicitly in [7]. However, it is known that if there exists a nonmetrizable normal Moore space, then there exists a countably paracompact nonnormal Moore space. [5].) Subsequently M. L. Wage constructed a first countable separable countably paracompact CWH (= collectionwise Hausdorff) zero-dimensional space with ω_1 points that fails to be normal, without using any axioms beyond ZFC, [6]. The purpose of this note is to show that the existence of such a space, which in addition is locally compact, follows in a natural way from the existence of a Hausdorff Gap, [2] (see the Lemma below). It seems to me that this construction, including the construction of a Hausdorff Gap, is easier than the construction in [6]. The fact that our construction is related in such a natural way to a classical combinatorial result makes it of independent interest.

Example. There exists a first countable separable countably paracompact CWH locally compact zero dimensional space that fails to be normal.

To explain what a Hausdorff Gap is, we need the following concept. If A and B are sets, then A is almost contained in

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B,A \subseteq * B, if A - B is finite; we write A \subset * B if A \subseteq * B but not B \subset * A.

Lemma. (Hausdorff, [2]) There are families $\{A(i, \alpha): \alpha \in \omega_1, i = 0, 1\}$ of subsets of ω , the finite ordinals, such that

(1) $\alpha < \beta$ implies $A(i, \alpha) \subset * A(i, \beta)$, i = 0, 1,

(2) $A(0,\alpha)$ ($A(1,\beta)$ is finite for all $\alpha,\ \beta\in\omega_1,$ and

(3) There is no $S \subset \omega$ such that $A(0, \alpha) \subset * S$ and

 $A(1,\alpha) \subset * \omega - S \text{ for all } \alpha \in \omega_1.$

(Actually Hausdorff considered $\{A(0,\alpha): \alpha \in \omega_1\}$ and $\{\omega - A(1,\alpha): \alpha \in \omega_1\}$ which explains the name Hausdorff Gap a little bit better.)

It is quite remarkable that this does not require the Continuum Hypothesis.

Construction of the Example

Let the A(i, α)'s be as in the Lemma. For i = 0,1 let $L_i = \{i\} \times (\omega_1 - \{0\})$. Topologize $H = L_0 \cup L_1 \cup \omega$ as follows. Points of ω are isolated, and a basic neighborhood of a point (i, β) $\in L_0 \cup L_1$ has the form

 $B(i,\beta,\alpha,F) = \{(i,\xi): \alpha < \xi \leq \beta\} \cup ((A(i,\beta) - A(i,\alpha)) - F)$ with $\alpha < \beta$ and $F \subset \omega$ finite. It is easily verified that this definition is correct, and that H is a first countable Hausdorff space. Since $A(i,\beta) - A(i,\alpha)$ is infinite whenever $\alpha < \beta$, the set ω is dense, hence H is separable.

In order to show that H is locally compact and zero-dimensional we show that each B(i, β, α, F) is compact. (Zero-dimensionality can be proved directly, of course.) Let \mathfrak{A} be an open cover of some B(i, β, α, F). For some $n \ge 1$ there are β_0, \dots, β_n , with $\beta = \beta_0 > \dots > \beta_n = \alpha$, and a finite G $\subset \omega$, such that $\mathfrak{B} = \{B(i, \beta_k, \beta_{k+1}, G): 0 \le k < n\}$ covers $\{(i, \xi): \alpha < \xi \le \beta\}$ and refines \mathfrak{A} . But then

$$\texttt{B}(\texttt{i}, \texttt{\beta}, \texttt{\alpha}, \texttt{F}) \ - \ \texttt{U} \ \mathscr{B} \ \subseteq \ \texttt{G} \ \texttt{U} \ \texttt{[(A(\texttt{i}, \texttt{\beta}_0) \ - \ \texttt{A(\texttt{i}, \beta_n))) \ -} \ \texttt{A(\texttt{i}, \beta_n))} \ - \ \texttt{A(\texttt{i}, \beta_n))} \ - \ \texttt{A(\texttt{i}, \beta_n))} \ - \ \texttt{A(\texttt{i}, \beta_n)} \ + \ \texttt{A(\texttt{i},$$

 $\bigcup_{0 \le k \le n} (A(i, \beta_k) - A(i, \beta_{k+1}))] = G$

since $(A_0 - A_n) - \bigcup_{0 \le k \le n} (A_k - A_{k+1}) = \emptyset$ for any sets A_0, \dots, A_n , so $B(i, \beta, \alpha, F)$ is compact since G and \mathcal{B} are finite.

The subspaces L_0 and L_1 of H are copies of $\omega_1 - \{0\}$, hence are countably compact. It follows that every countable open cover of H contains a finite subcollection which covers the nonisolated points, hence H is countably paracompact, and it also follows that each closed discrete subset of H consisting of nonisolated points is finite, hence H is CWH.

It remains to show that H is not normal. Assume the contrary. Then the disjoint closed sets L_0 and L_1 would have disjoint neighborhoods, U_0 and U_1 say. For each $\beta \in \beta_1 - \{0\}$ we can choose $f(\beta) < \beta$ and a finite $E(\beta) \subset \omega$, not depending on i (this only is a trick), such that $B(i,\beta,f(\beta),E(\beta)) \subseteq U_i$ for i = 0,1. Since $f(\beta) < \beta$ for all $\beta \in \omega_1 - \{0\}$ there is, as well known, a $\kappa \in \omega_1$ and a cofinal $K \subset \omega_1$ such that $f(\beta) = \kappa$ for $\beta \in K$. Define S_0 and S_1 by

 $\mathbf{S}_{\mathbf{i}} = [\mathbf{A}(\mathbf{i},\kappa) \ \bigcup \ (\omega \Pi \mathbf{U}_{\mathbf{i}})] - \mathbf{A}(1 - \mathbf{i},\kappa).$

Then $S_0 \cap S_1 = \phi$ since $U_0 \cap U_1 = \phi$. Let i = 0, 1 and $\alpha \in \omega_1$ be arbitrary. Since K is cofinal, there is a $\beta \in K$ with $\beta \geq \alpha$, hence with $A(i,\alpha) \subseteq * A(i,\beta)$. But $f(\beta) = \kappa$, hence $(A(i,\beta) - A(i,\kappa)) - E(\beta) \subseteq \omega \cap U_i$, so $A(i,\beta) \subseteq * A(i,\kappa) \cup (\omega \cap U_i)$. Since $A(i,\alpha) \cap A(1-i,\kappa)$ is finite, it follows that $A(i,\alpha) \subseteq * S_i$. Since i = 0, 1 and $\alpha \in \omega_1$ are arbitrary, and $S_0 \cap S_i = \phi$ this contradicts (3) of the Lemma.

This completes the verification of the properties of H.

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