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# A NOTE ON PREPARACOMPACTNESS

by
J. C. Smith

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**Web:** http://topology.auburn.edu/tp/
**Mail:** Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
**E-mail:** topolog@auburn.edu
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## 1. Introduction

In 1973 R. C. Briggs [5] introduced two properties, preparacompactness (ppc) and $\aleph$-preparacompactness ($\aleph$-ppc) and compared them with the properties of paracompactness and collectionwise normality in various q-spaces. The purpose of this paper is to show that most of the results obtained in [5] can be generalized, hence closing the somewhat large gap between these properties.

*Definition 1.1* A $T_2$ space X is *preparacompact (resp. $\aleph$-preparacompact)* if each open cover of X has an open refinement $\mathcal{H} = \{H_\alpha: \alpha \in A\}$ such that, if $B \subseteq A$ is infinite (resp. uncountable) and if $p_\beta$ and $q_\beta \in H_\beta$ for each $\beta \in B$ with $p_\alpha \neq p_\beta$ and $q_\alpha \neq q_\beta$ for $\alpha \neq \beta$, then the set $Q = \{q_\beta: \beta \in B\}$ has a limit point whenever $P = \{p_\beta: \beta \in B\}$ has a limit point. The notions of *$\sigma$-ppc* and *$\sigma$-$\aleph$-ppc* should be clear. Collections satisfying the above property will be called *ppc($\aleph$-ppc) collections.*

Since neither of the above properties implies paracompactness, even in the presence of collectionwise normality, the special setting of q-spaces is chosen for their study.

*Definition 1.2* A space X is called a *q-space* if each point $p \in X$ has a sequence of neighborhoods $\{N_i\}_{i=1}^\infty$ such that if $y_i \in N_i$ for each i with $y_i \neq y_j$ for $i \neq j$, then the set $\{y_i\}_{i=1}^\infty$ has a limit point.

In [5] Briggs obtained the following.

*Theorem 1.3 Let* X *be a regular* q-*space. Then the following are equivalent:*

*(1)* X *is paracompact.*

*(2)* X *is* $\aleph$*-ppc and subparacompact.*

*(3)* X *is* $\aleph$*-ppc and metacompact.*

Since the notion of a $\theta$-refinability of J. Worrell and H. Wicke [10] is a generalization of both subparacompactness and metacompactness, it is natural to ask whether the above result can be generalized accordingly. In §2 of this paper we actually obtain a much stronger result using the notion of irreducible spaces [6]. Theorems involving the properties of $\delta\theta$-refinability [1] and weak $\overline{\delta\theta}$-refinability [9] are obtained in §3, and in §4 it is shown that every $\sigma$-$\aleph$-ppc, normal q-space is collectionwise normal. Examples and open questions are also included in §4.

## 2. Irreducible q-spaces

*Definition 2.1* An open cover $\mathcal{G}$ of a topological space X is called *minimal* provided no proper subcollection of $\mathcal{G}$ covers X. A space X is called *irreducible* if every open cover of X has a minimal open refinement.

The following lemmas are easy to verify and hence the proofs are omitted.

*Lemma 2.2 Let* $\mathcal{G} = \{G_\alpha: \alpha \in A\}$ *be an open cover of an irreducible space* X. *Then* $\mathcal{G}$ *has a minimal refinement* $\mathcal{H} = \{H_\beta: \beta \in B\}$ *where* $H_\beta \subseteq G_\beta$ *for all* $\beta \in B \subseteq A$.

*Lemma 2.3 A cover* $\mathcal{W} = \{W_\alpha: \alpha \in A\}$ *is a minimal cover of* X *iff there exists a discrete collection of non-empty closed sets* $\{F_\alpha: \alpha \in A\}$ *such that* $F_\alpha \subseteq W_\alpha$ *for each* $\alpha \in A$.

*Theorem 2.4 Let* X *be a* q*-space and let* $\mathcal{G} = \{G_\alpha: \alpha \in A\}$ *be a* $\aleph$*-ppc collection of open subsets of* X. *If there exists a discrete collection* $\{D_\beta: \beta \in B\}$ *of non-empty subsets of* X *such* *that* $D_\beta \subseteq G_\beta$ *for each* $\beta \in B \subseteq A$, *then* $\{G_\beta\colon \beta \in B\}$ *is either countable or locally finite.*

*Proof:* Suppose B is uncountable and $\{G_\beta\colon \beta \in B\}$ is not locally finite at $p \in X$. Since X is a q-space, there exists a countable subcollection $\{G_{\beta_i}\}_{i=1}^{\infty}$ of $\mathcal{G}$ and a sequence of points $\{p_i\}_{i=1}^{\infty}$ such that

(i) $p_i \in G_{\beta_i}$ for each i,

(ii) $p_i \neq p_j$ and $G_{\beta_i} \neq G_{\beta_j}$ for $i \neq j$,

(iii) $\{p_i\}_{i=1}^{\infty}$ has a limit point in X.

Now let $q_\beta \in D_\beta$ for each $\beta \in B$ and define $p_\beta = q_\beta$ for all $\beta \notin \{\beta_i\colon i=1,2,\cdots\}$. Then $P = \{p_\beta\colon \beta \in B\}$ has a limit point while $Q = \{q_\beta\colon \beta \in B\}$ does not. This contradicts the fact that $\mathcal{G}$ is an $\aleph$-ppc collection. Hence $\{G_\beta\colon \beta \in B\}$ is locally finite.

*Remark:* If $\aleph$-ppc is replaced by ppc in the above theorem then $\{G_\beta\colon \beta \in B\}$ is locally finite in each case.

*Theorem 2.5 Let* X *be a regular* q-*space. Then* X *is paracompact iff* X *is* $\aleph$*-ppc and irreducible.*

*Proof:* The necessity is clear. Let X be $\aleph$-ppc and irreducible and let $\mathcal{U}$ be any open cover of X. Then $\mathcal{U}$ has an open $\aleph$-ppc refinement $\mathcal{G} = \{G_\alpha\colon \alpha \in A\}$. Since X is irreducible $\mathcal{G}$ has an open refinement $\mathcal{H}$ which covers X minimally. By Lemma 2.2 above we may assume that $\mathcal{H} = \{H_\beta\colon \beta \in B\}$ where $H_\beta \subseteq G_\beta$ for each $\beta \in B \subseteq A$. By Lemma 2.3 there exists a discrete collection of non-empty closed sets $\{D_\beta\colon \beta \in B\}$ such that $D_\beta \subseteq H_\beta$ for each $\beta \in B$. Therefore, $\{G_\beta\colon \beta \in B\}$ is a $\sigma$-locally finite open refinement of $\mathcal{U}$, and hence X is paracompact by Theorem 1 of [7].

*Corollary 2.6 Let* X *be a* q-*space. Then* X *is paracompact iff* X *is ppc and irreducible.*

*Proof:* The proof follows immediately from the remark after Theorem 2.4 above.

*Corollary 2.7* *Let* X *be a regular* q-*space. Then the following are equivalent:*

*(1)* X *is paracompact.*

*(2)* X *is* $\aleph$-*ppc and* $\theta$-*refinable.*

*(3)* X *is* $\aleph$-*ppc and weak* $\overline{\theta}$-*refinable.*

*Proof:* In [9] the author has shown that $\theta$-refinable and weak $\overline{\theta}$-refinable spaces are irreducible.

*Remark:* It should be noted at this point that the above results (assuming regularity) remain true when $\aleph$-ppc is replaced by $\sigma$-$\aleph$-ppc by Theorem 2.4.

## 3. $\delta\theta$-refinable Spaces

In [1] Aull proved that $\aleph_1$-compact $\delta\theta$-refinable spaces are Lindelöf and in [8] the author obtained an analogous result for weak $\overline{\delta\theta}$-refinable spaces.

*Definition 3.1* A space X is called $\delta\theta$-*refinable* if every open cover X has a refinement $\mathcal{G} = \bigcup_{i=1}^{\infty} \mathcal{G}_i$ satisfying,

(i) each $\mathcal{G}$ is an open cover of X.

(ii) for each $x \in X$ there exists an integer $n(x)$ such that $\operatorname{ord}(x, \mathcal{G}_{n(x)}) \leq \aleph_o$.

*Definition 3.2* A space X is called weak $\overline{\delta\theta}$-*refinable* if every open cover of X has a refinement $\mathcal{G} = \bigcup_{i=1}^{\infty} \mathcal{G}_i$ satisfying,

(i) each $\mathcal{G}_i$ is a collection of open subsets of X.

(ii) for each $x \in X$ there exists an integer $n(x)$ such that $0 < \operatorname{ord}(x, \mathcal{G}_{n(x)}) \leq \aleph_o$.

(iii) $\{G_i^* = \bigcup\{G: G \in \mathcal{G}_i\}\}_{i=1}^{\infty}$ is point finite.

Even though $\delta\theta$-refinable spaces need not be irreducible it is natural to ask whether similar results to those in §2 can be obtained since such spaces are generalizations of $\theta$-refinable spaces. Here we provide such results using the notion of maximal distinguished sets, due to Aull [1].

Let $\mathcal{U}$ be an open cover of a topological space X.

*Definition 3.3* A set M is *distinguished* with respect to $\mathcal{U}$ if for each pair $x, y \in M$ with $x \neq y$, then $x \in U \in \mathcal{U} \Rightarrow y \notin U$.

*Lemma 3.4 For every subset* M *of a space* X *and every open (in* X*) cover* $\mathcal{U}$ *of* M*, there exists a maximal distinguished set with respect to* $\mathcal{U}$ *which is discrete in* $\bigcup\{U: U \in \mathcal{U}\}$.

*Theorem 3.5 Let* X *be a regular* q-*space. Then* X *is paracompact iff* X *is* $\aleph$*-ppc and* $\delta\theta$*-refinable.*

*Proof:* Let X be $\aleph$-ppc and $\delta\theta$-refinable and let $\mathcal{U}$ be an open cover of X. Then $\mathcal{U}$ has an $\aleph$-ppc refinement $\mathcal{G} = \{G_\alpha: \alpha \in A\}$. Since X is $\delta\theta$-refinable, $\mathcal{G}$ has a refinement $\bigcup_{i=1}^{\infty}\mathcal{W}_i$ satisfying,

(i) each $\mathcal{W}_i = \{W(\alpha,i): \alpha \in A\}$ is an open cover of X,

(ii) for each $x \in X$, there exists an integer $n(x)$ such that $\operatorname{ord}(x, \mathcal{G}_{n(x)}) \leq \aleph_0$.

As before we may assume $W(\alpha,i) \subseteq G_\alpha$ for each $\alpha \in A$ and each i. Now let $H_n = \{x: \operatorname{ord}(x, \mathcal{G}_n) \leq \aleph_0\}$ so that $X = \bigcup_{n=1}^{\infty}H_n$. Let $M_n$ be a maximal distinguished set of $H_n$ with respect to $\mathcal{G}_n$ for each n. By Lemma 3.4 the collection of singletons of points of each $M_n$ is a discrete collection in X. By Theorem 2.4 above $H_n$ is covered by a $\sigma$-locally finite subcollection of $\mathcal{W}_n$ for each n. Therefore $\mathcal{U}$ has a $\sigma$-locally finite open refinement, and hence X is paracompact.

The analogous result for weak $\overline{\delta\theta}$-refinable spaces is also true. The proof is a modification of the one above and hence is omitted.

*Theorem 3.6 Let* X *be a regular* q-*space. Then* X *is paracompact iff* X *is* $\aleph$*-ppc and weak* $\overline{\delta\theta}$*-refinable.*

**4. Normal-q-spaces**

In [5] Briggs obtained the following result using a somewhat involved argument. We now generalize this result using a theorem of Zenor [11].

*Theorem 4.1* (Briggs) *Let* X *be a normal* q-*space. If* X *is* $\aleph$-*ppc, then* X *is collectionwise normal.*

*Theorem 4.2* (Zenor) *A space* X *is collectionwise normal iff for each discrete collection* $\{F_\alpha: \alpha \in A\}$ *of closed sets, there exists a sequence of collections* $\{V(\alpha,i): \alpha \in A\}_{i=1}^{\infty}$ *of open subsets of* X *satisfying,*

*(i)* $\{V(\alpha,i)\}_{i=1}^{\infty}$ *covers* $F_\alpha$ *for each* $\alpha \in A$,

*(ii)* $F_\alpha \cap [\bigcup_{\beta\neq\alpha} V(\beta,i)]^{-} = \emptyset$ *for each* $\alpha \in A$ *and each* i.

*Theorem 4.3* *Let* X *be a normal* q-*space. If* X *is* $\sigma$-$\aleph$-*ppc, then* X *is collectionwise normal.*

*Proof:* Let $\{F_\alpha: \alpha \in A\}$ be an uncountable discrete collection of closed subsets of X. Since X is normal there exists for each $\alpha \in A$ an open set $G_\alpha$ containing $F_\alpha$ such that $\overline{G}_\alpha \cap [\bigcup_{\beta\neq\alpha} F_\beta] = \emptyset$. We may assume that $0 \notin A$. Then let $G_0 = X - [\bigcup_{\alpha\in A} F_\alpha]$, and $\mathcal{G} = \{G_\alpha: \alpha \in A\} \cup \{G_0\}$. Since X is $\sigma$-$\aleph$-ppc, $\mathcal{G}$ has a refinement $\bigcup_{i=1}^{\infty} \mathcal{H}_i$ where $\mathcal{H}_i = \{H(\alpha,i): \alpha \in A\}$ has the $\aleph$-ppc property and $H(\alpha,i) \subseteq G_\alpha$ for each $\alpha \in A$ and each i. Let $\mathcal{H}_i^* = \{H(\alpha,i): H(\alpha,i) \cap F_\alpha \neq \emptyset\}$ for each i. Then by Theorem 2.4, each $\mathcal{H}_i^*$ is either countable or locally finite so that $\{H(\alpha,i): \alpha \in A\}_{i=1}^{\infty}$ satisfies the conditions of Theorem 4.2 above. Therefore X is collectionwise normal.

Briggs [5] used several examples to demonstrate the necessity of a special setting (q-spaces) in order to study the relationships between preparacompact spaces and other more common generalizations of paracompactness. These examples are summarized here for the benefit of the reader. For more details see [5].

*Example I:* A countably compact, first countable, normal q-space which is ppc and collectionwise normal but not paracompact.

*Example II:* A first countable, collectionwise normal q-space which is not $\aleph$-ppc.

*Example III:* A normal, metacompact, ppc space which is not collectionwise normal.

*Example IV:* A regular, locally countably compact q-space which is $\aleph$-ppc and $\sigma$-ppc but not ppc.

*Example V:* A regular, countably compact, q-space which is ppc but not normal.

*Example VI:* A metacompact, first countable, Lindelöf q-space which is $\aleph$-ppc but not regular.

Several interesting open questions remain:

(1) Is every regular, first countable, ppc space normal?

(2) Is Theorem 3.5 true for weak $\theta$-refinable spaces?

(3) In what setting, other than q-spaces, are the above results true?

(4) When are ppc spaces expandable?

(5) When are $\aleph$-ppc spaces countably paracompact?

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Virginia Polytechnic Institute and State University

Blacksburg, Virginia 24061