INDUCED INVOLUTIONS ON HILBERT CUBE HYPERSPACES

by

J. E. West

INDUCED INVOLUTIONS ON HILBERT
CUBE HYPERSPACES [1]

J. E. West [2]

1. Introduction

If the Hilbert cube, Q, is represented as the Cartesian product, $\prod_{i=1}^{\infty} J_i$, of closed intervals $J_i = J = [-1,1]$, then the natural reflection $\sigma(t) = -t$ of J about its mid-point induces a family of reflections of Q by independent (diagonal or product) action on various coordinates. These are so natural that they should be the standards against which all others must be compared. Two extreme cases are readily apparent: (a) σ_1, which reflects only J_1 and (b) σ_{∞}, which reflects all J_i simultaneously.

It is easy to see that some of the involutions of Q which leave precisely $Q_0 = \{q = (t_1, t_2, \cdots) \in Q \mid t_1 = 0\}$ fixed are topologically equivalent to σ_1, namely those which actually reflect in the first coordinate[*]; for, if $Q_- = \{q = (t_1, t_2, \cdots) \in Q \mid t_1 \leq 0\}$ and $Q_+ = \{q = (t_1, t_2, \cdots) \in Q \mid t_1 \geq 0\}$, then for any such "reflective" involution T, i.e. $T(Q_+) = Q_-$, we can define a homeomorphism $S: Q \to Q$ to be the identity on Q_+ and to be $\sigma_1 \cdot T^{-1} \mid Q_-$ on Q_-. Then $\sigma_1 = STS^{-1}$.

The situation in respect to involutions with a single fixed point is much less clear. In particular, it is not known at all whether there exists an involution on Q with unique

---

[*] This need not be necessary, e.g., let $\sigma_2 \colon Q \to Q$ reflect in precisely the odd-indexed coordinates. Then $Q_e = \{q \in Q \mid q = (0, t_2, 0, t_4, 0, \cdots)\}$ is the fixed point set of σ_2, is a Z-set and hence (by a theorem of R. D. Anderson [1]) the *pair* (Q, Q_e) is homeomorphic to (Q_-, Q_0) as well as to (Q_+, Q_0). There is, then, an involution induced on Q with fixed point set Q_0 but which does not reflect. These, of course, cannot be conjugate to σ_1. Are they conjugate to one another? (Probably not.)

fixed point which is not "standard" in the sense of being con-
jugate to the refelction σ_∞. The best, indeed, the only, pub-
lished result on this question is by R. Wong [14], who proved
that an involution T of Q with exactly one fixed point is
standard precisely when the fixed point has a basis of T-
invariant, contractible neighborhoods.

Since the question of whether all involutions of Q with
single fixed point satisfy this condition is quite difficult
(it leads, via work of Chapman and Siebenmann, straight to a
murky area of proper homotopy theory and shape theory--the
question of "Whitehead" theorems for infinite-dimensional ob-
jects), the question of just how esoteric an involution one can
dream up is quite intriguing, especially if one asks for an
essentially infinite dimensional construction, rather than a
product of finite-dimensional actions.

One of the most esoteric constructions of Hilbert cubes is
the formation of Hyperspaces. Consider the involution $\sigma: J \to J$
above. Look at 2^J, the hyperspace of all non-void, closed sub-
sets of J under the Hausdorff metric, ρ. [$\rho(A,B) = \min\{\varepsilon > 0 \mid$
$A \subseteq N_\varepsilon(B)$ and $B \subseteq N_\varepsilon(A)\}$, where $N_\varepsilon(A)$ is the set of all points
t which are within ($\leq$) ε of some point of A]. Now, 2^J is a
Hilbert cube ([11]). (In fact, whenever X is a non-degenerate
Peano continuum, 2^X is a Hilbert cube ([6],[7],[8]). Moreover,
the hyperspace construction is functorial, i.e., if f: X → Y is
a mapping, then $2^f: 2^X \to 2^Y$ defined by $2^f(A) = f(A)$ is also
continuous. So, we have an involution $2^\sigma: 2^J \to 2^J$. What is
it like? The first thing we notice is that its fixed point set,
$\mathfrak{S}$, is all symmetric closed subsets of J and is enormous. In
fact, if we let J/σ be the space of σ-orbits, then this is just
homeomorphic to [0,1], and the orbit mapping p: J → J/σ = [0,1]
is simply the absolute value. Now, $\mathfrak{S}$ is easily seen to be
homeomorphic to 2^I (see Lemma 1) and to have Property Z in 2^X

(see the preface to Lemma 3 for definition and the initial

portion of the proof for the case under discussion here).

Therefore, $2^J/\mathcal{S}$ is a Hilbert cube ([4]). The involution 2^σ

now induces an involution $\bar{\sigma}$ on $2^J/\mathcal{S}$ which has exactly one fixed

point, $\mathcal{S}$, so we have constructed from σ a *new* involution

with single fixed point on a Hilbert cube,...or have we?

Consider the function N: $2^J \times [0,2] \to 2^J$ given by

$N(A,t) = N_t(A)$ (as defined earlier, the closed t-neighborhood

of A in J). This is an equivariant contraction of 2^J to J

which preserves $\mathcal{S}$. Therefore it induces an equivariant con-

traction $\bar{N}$ of $2^J/\mathcal{S}$. The neighborhoods $U_\varepsilon = \{A \in 2^J | \rho(A,S) \leq \varepsilon$

for some $S \in \mathcal{S}\}$ are contracted in themselves by N and thus

their images $U_\varepsilon/\mathcal{S}$ in $2^J/\mathcal{S}$ satisfy the hypotheses of Wong's

theorem, so $\bar{\sigma}$ is standard. (That is, there is a homeomorphism

h: $2^J/\mathcal{S} \to Q$ with $h(\mathcal{S}) = (0,0,\cdots)$ such that $h\bar{\sigma}h^{-1} = \sigma_\infty$.)

We might have expected this, since the system (J,σ) is

so simple, but is it possible that somewhere there is a Peano

continuum X with involution T such that $2^X/\mathcal{S}$ is a Hilbert cube

and the induced involution $\bar{T}$ is not standard? The answer,

which is the theorem of this paper, is no(!)

Theorem: Let X be a non-degenerate Peano continuum and

T: X → X an involution.   Then the hyperspace $\mathcal{S}$ of T-symmetric

closed subsets of X is a Hilbert sub-cube of 2^X which is a

Z-set if (and only if) the fixed point set of T is nowhere

dense in X.   In this case, the induced involution

$\bar{T}$: $2^X/\mathcal{S} \to 2^X/\mathcal{S}$ is standard.

2. Proofs

Here we give a sequence of lemmas culminating in a proof

of the announced theorem, the outline of which has already been

presented.

Lemma 1: If T *is an involution on a non-degenerate Peano*
continuum X, *then the induced involution* 2^T *on the hyperspace*
2^X *has a fixed point set* $\mathfrak{S}$ *which is a Hilbert sub-cube of* 2^X.

Proof: Let p: X → X/T be the orbit mapping. Because the
orbits of T form a continuous decomposition of X, the function
p^{-1}: $2^{X/T}$ → 2^X is continuous. It is one-to-one, and its image
is $\mathfrak{S}$. Since X/T is also a Peano continuum, $2^{X/T}$ is a Hilbert
cube ([6],[7],[8]), and, thus, so is $\mathfrak{S}$.

Lemma 2: $\mathfrak{S}$ *is nowhere dense in* 2^X *if and only if the fixed*
point set, F, *of* T *is nowhere dense in* X.

Proof: As $\mathfrak{S} \supset 2^F$, and 2^F contains an open set of 2^X if
F contains one of X, it is clear that $\mathfrak{S}$ is nowhere dense only
if F is. On the other hand, if F is nowhere dense and S ∈ $\mathfrak{S}$,
then we may either add to S a single point of X\(S ∪ F) which
is close to S, thus unbalancing S by a small perturbation, or, in
the event that F separates S from the remainder of X, select
a point s ∈ S\F, and remove from S a small open neighborhood N
of s which misses its T-image, also unbalancing S by a small
perturbation. In either case, we see that $\mathfrak{S}$ is not dense at S.

A closed subset, A, of an absolute neighborhood retract,
B, has *Property* Z *in* B if it is nowhere dense and if each map-
ping of an n-cube into B is uniformly approximable by mappings
of the n-cube into B\A. (See [2],[1],[9],[5].)

Lemma 3: If F *is nowhere dense in* X, *then* $\mathfrak{S}$ *has Property*
Z *in* 2^X.

Proof: The idea of the proof is simple and quite clearly
illustrated in the case of the arc J and involution σ: let
ε > 0 be given. Let f_1: J → J be multiplication by (1-ε/2),
let f_2: 2^J → 2^J be the mapping $N_{\varepsilon/4}$ sending each set A to its
closed ε/4-neighborhood, and let f_3: $2^{[-1+\varepsilon/4, 1-\varepsilon/4]}$ → 2^J be

the mapping defined by $A \to A \cup \{\min(A) - \varepsilon/4\}$. Then
$f_3 \circ f_2 \circ 2^{f_1}: 2^J \to 2^J \setminus S$ (because $f_3 \circ f_2 \circ 2^{f_1}(A)$ has precisely one
isolated point which is its minimum, and thus cannot be sym-
metric), and is within ε of the identity, so S has Property Z.

To push the above argument to the general case is quite
a bit more complicated, the problem being how to unbalance the
sets in a continuous manner, as there is in general no canonical
way of choosing a point from each set (done above by taking the
minimum value). Moreover, if the metric on X is not convex,
then the "neighborhood function" applied above is in general
not continuous. Fortunately, Bing [3] has proved that each
Peano continuum admits a convex metric, d, which we assume is
the given one. (A metric is *convex* if each pair of points a,
b has a mid-point, that is, a point c such that $d(a,c) = d(c,b) = $
$1/2\ d(a,b)$. There may be more than one, but this concept assures
the existence of "geodesic" arcs between any two points.) To
reduce the problem of finding the analogues of "end points,"
such as min(A) in the example, we employ the machinery of Curtis
and Schori ([6],[7],[8]), who, taking an idea of J. Segal's,
showed that if X is any Peano continuum, then 2^X is homeomorphic
to an inverse limit $\varprojlim \{2^{\Gamma_i}, f_i\}$ of hyperspaces of embedded
graphs Γ_i in such a way that there are maps $p_i: 2^X \to 2^{\Gamma_i}$ which
converge uniformly to the identity as $i \to \infty$.

We are now ready for the general case of the lemma. Let
X be any non-degenerate Peano continuum with involution T
having nowhere dense fixed point set F. Let d be a convex
metric for X and also denote the Hausdorff metric on 2^X derived
from it. Let $\{\Gamma_i\}_{i=1}^{\infty}$ be a sequence of finite embedded graphs
as above.

Let $\alpha: I^n \to 2^X$ be any mapping, and fix $\varepsilon > 0$. Let
$\alpha' = N_{\varepsilon/3} \circ \alpha: I^n \to 2^X$ and note that α' misses 2^F. Choose i so
large that the distance from p_i to the identity is less than

$\varepsilon/3$ and the minimum distance between points of the image of α'
and points of 2^F, and let $\alpha'' = p_i \cdot \alpha': I^n \to 2^{\Gamma_i}$. Note that
$\alpha''(I^n)$ misses $2^F \cap 2^{\Gamma_i}$ and α'' is within $2\varepsilon/3$ of α.

To complete the proof, we need only find a map
$\beta: I^n \to 2^{\Gamma_i} \setminus \mathcal{S}$ which is within $\varepsilon/3$ of α''. We may do this if
we show that $(2^{\Gamma_i} \cap \mathcal{S}) \setminus 2^F$ is a Z-set in $2^{\Gamma_i} \setminus 2^F$, and, since
$2^{\Gamma_i} \setminus 2^F$ is complete, it is immediate from the given definition
of Property Z that we need only show $(2^{\Gamma_i} \setminus 2^F) \cap \mathcal{S}$ to be a
countable union of Z-sets of $2^{\Gamma_i} \setminus 2^F$.

Let ρ be a convex metric for Γ_i, denoting also by ρ the
Hausdorff metric for 2^{Γ_i} derived from it. For the rest of this
proof, distances will be measured by ρ unless otherwise spe-
cified. Also, the neighborhood function N_t refers henceforth
"ρ-neighborhoods in Γ_i."

Now let S be any T-symmetric subset of Γ_i not lying in F.
We consider three cases: (a) S has a boundary (in Γ_i) point
which is not in F and is not a branch point of Γ_i, (b) each
boundary point of S is a branch point of Γ_i, and (c) each
boundary point of S lies in F.

In case (a) we have a situation somewhat analogous to the
(J,σ) example. We find a closed neighborhood of S which misses
2^F and intersects $\mathcal{S}$ in a Z-set of 2^{Γ_i}. Let $\delta_1 > 0$ be small
enough that $N_{\delta_1}(S) \cap 2^F = \emptyset$ and $N_{\delta_1}(S) \cap T(N_{\delta_1}(S)) = \emptyset$. Now
choose a boundary point x of S which is not in F and is not a
branch point of Γ_i and let $\delta_2 > 0$ be less than or equal to δ_1
and such that $N_{\delta_2}(x)$ misses Γ_i's branch points. Pick $c \in N_{\delta_2}(x)$
such that the open arc (t,c) joining x to c in $N_{\delta_2}(x)$ contains
no point of S, and let $\delta_3 = \rho(x,c)$, denoting by c' the point of
(x,c) at distance $\delta = \delta_3/3$ from x. We show that for any $\delta' < \delta$,
$\mathcal{S} \cap N_{\delta'}(S)$ is a Z-set of 2^{Γ_i} by showing that for any $\varepsilon' > 0$
there is a mapping $\gamma: 2^{\Gamma_i} \to 2^{\Gamma_i}$ which misses $\text{int}(N_\delta(S)) \cap \mathcal{S}$
and is within ε' of the identity. The method is first to

"fatten" and then to "unbalance" by introducing isolated points:
Let $\gamma_1: 2^{\Gamma_i} \to 2^{\Gamma_i}$ be defined by $\gamma_1(A) = N_{\theta(A)}(A)$, where $\theta(A) =$
$(\varepsilon'/2)\max\{0, \delta - d(A,S)\}$. Now θ is continuous, and the neighborhood
function is a homotopy, so γ_1 is continuous, is within $\varepsilon'/2$
of the identity, is the identity off $N_\delta(S)$, and carries $N_\delta(S)$
into itself. Moreover, if $A \in \text{int}(N_\delta(S))$, then $\gamma_1(A)$ has no
isolated point. Additionally, if $A \in \text{int}(N_\delta(S))$, then $\gamma_1(A)$
has a unique point nearest c', and this point lies in $N_\delta(x)$
and varies continuously with A as A varies within $N_\delta(S)$. (We
have been at great pains to produce a situation in which this
"nearest point" function is continuous.) Now let, for $A \in N_\delta(S)$,
m(A) be the nearest point of A to c', and let a(A) be the point
of (c',m(A)) at distance $(\varepsilon'/2)(\delta - d(A,S))$ from m(A). Then
$\gamma_2: N_\delta(S) \to N_\delta(S)$ given by $\gamma_2(A) = A \cup \{a(A)\}$ is within $\varepsilon'/2$
of the identity, adds an isolated point to each $A \in \text{int}(N_\delta(S))$,
and is the identity on the boundary of $N_\delta(S)$. Extend γ_2 to all
of 2^{Γ_i} by the identity, and let $\gamma = \gamma_2 \circ \gamma_1: 2^{\Gamma_1} \to 2^{\Gamma_1}$. This is
the promised function: clearly within ε' of the identity, it
misses $\mathfrak{S} \cap \text{int}(N_\delta(S))$ because if $\gamma(A) \in \text{int}(N_\delta(S))$, then $\gamma(A)$
contains precisely one isolated point and that point is not in
F.

Case (b) comprises at most finitely many members of 2^{Γ_i}.
As 2^{Γ_i} is a Hilbert cube, each member of it has Property Z in
it. (The Hilbert cube is homogeneous [5], and a point of
$\prod\limits_{i=1}^{\infty} J_i$ which has some coordinate equal to ± 1 is clearly a Z-set.)

Case (c) is handled as follows: Let the components of
$\Gamma_i \backslash F$ be $C_1, C_2, \cdots$. Discard all C_i such that TC_i is not a C_j.
For each remaining one, let $K_i = \bar{C}_i \cup T(C_i)$ and reindex them
$K_1, K_2, \cdots$. As the boundary of S lies entirely within F, then
because $S \not\subset F$, S is the union of $S \cap F$ and a collection of K_i's.
We show that if $\mathcal{K}_j = \{A \in 2^{\Gamma_i} | K_j \subset A\}$ then $\mathcal{K}_j$ is a Z-set of
2^{Γ_i}.

This is done, though not so labelled, in Lemma 4.2 of [12], which asserts that for any point z of a graph Γ, the subcontinua of Γ containing z form a Z-set in 2^Γ. The proof actually is by means of a locally determined deformation and shows that the closed subsets of Γ which contain non-degenerate subcontinua passing through z form a Z-set in 2^Γ. We repeat it here. Let $z \in K_j \backslash F$ be a non-branch point. As it is a local cut point of $K_j \backslash F$, we may pick two points b_1 and b_2 on either side of z in its component of $K_j \backslash (F \cup$ branch points of $\Gamma_i)$. For simplicity, assume each of b_1 and b_2 is at distance 1 from z and parametrize the arcs $[z,b_k]$ by distance from z, letting $[t_k]$ be the point of $[z,b_k]$ at distance t_k from z, if $0 \le t_k \le 1$.

Now, fixing $\xi > 0$, let for each $A \in 2^{\Gamma_i}$, $[a_k]$ be the nearest point to z of $A \cap [z,b_k]$, if any, and for $0 \le a_k \le \xi$, let $a_k' = \max\{0, 2a_k - \xi\}$. Note that $a_k' = 0$ if $a_k \le \xi/2$ and $a_k' = a_k$ if $a_k = \xi$. Let $\delta = \delta(A) = \min\{\rho(a,z) | a \in A\}$ and set

$$f(A) = \begin{cases} A \cup \{[a_k'] | 0 \le a_k \le \xi, \ k = 1,2\}, & \text{if} \quad \delta \ge \xi/2 \\ A \cup \{[(2\delta/\xi)a_k' + (1-2\delta/\xi)a_k] | 0 \le a_k \le \xi, \ k = 1,2\}. & \text{if} \quad 0 \le \delta \le \xi/2 \end{cases}$$

Now f moves sets toward those sets which contain z, but no more than $\delta/2$, and at the same time creates isolated points toward z. The next function excises neighborhoods of z of varying radius from sets while adding their boundaries, thus effectively "discontinuing" all sets near z: Let $\alpha(A) = \max\{0, \xi/2 - \delta(A)\}$, and set

$$gf(A) = \begin{cases} (f(A)\backslash N_{\alpha(A)}(z)) \cup BdN_{\alpha(A)}(z), & \text{if} \quad \delta(A) < \xi/2 \\ f(A), & \text{if} \quad \delta(A) \ge \xi/2 \end{cases}$$

Then gf is within ξ of the identity, and its image misses $\mathcal{K}_j$, which must therefore be a Z-set.

Now $\mathcal{S} \cap (2^{\Gamma_i}\backslash 2^F)$ is covered by a countable collection of compact Z-set neighborhoods (in $\mathcal{S}$) of elements satisfying condition (a), together with a finite number of elements satisfying

condition (b) and a countable collection of Z-sets of the

form $S \cap (\mathcal{K}_j \backslash 2^F)$ (closed subsets of Z-sets are, of course, Z-

sets), so it is a Z-set in $2^{\Gamma_i} \backslash 2^F$ and the Lemma is proved (S

is a Z-set in 2^X).

   Lemma 4:   There is an invariant, convex metric on X.

   Proof: Let d be a convex metric on X/T, and let p: X → X/T

be the orbit map. We "pull d back" to an invariant, convex

metric on X in a standard manner: Let α: [a,b] → X/T be a path.

For each partition $\mathcal{P} = \{a = t_0 \leq t_1 \leq \cdots \leq t_n = b\}$ of [a,b], let

$\ell(\alpha, \mathcal{P}) = \sum_{i=1}^{n} d(\alpha(t_i), \alpha(t_{i-1}))$. Call α rectifiable if $\{\ell(\alpha, \mathcal{P})\}_{\mathcal{P}}$

is bounded above, and in this case let $\ell(\alpha)$ be its least upper

bound. Then set $\rho(x,y)$ to be the greatest lower bound of

$\{\ell(p \circ \alpha) | \alpha: ([a,b],a,b) \rightarrow (X,x,y)$ and $p \circ \alpha$ is rectifiable}. (We

remark that because d is a convex metric for X/T and p|X\F:

X\F → (X\F)/T is a covering map and hence has the unique path

lifting property (for a given lift of initial point), we may

restrict our collection of paths α to those with the property

that p∘α is a piecewise geodesic, i.e., for some partition $\mathcal{P}$ of

[a,b], $p \circ \alpha | [t_{i-1}, t_i]$ is an isometry.) Now we claim that ρ is

a convex metric for X. (As it is readily seen to be a metric,

we omit that verification.) Let x,y ∈ X. We need to produce a

mid-point, z, with $\rho(x,z) = \rho(z,y) = 1/2 \, \rho(x,y)$. For each

n = 1,2,···, let $\alpha_n: [a_n, b_n] \rightarrow X$ be a path such that $p \circ \alpha_n$ is

piecewise geodesic and $\ell(\alpha_n) < \rho(x,y) + 1/n$. Let $t_n \in (a_n, b_n)$

such that $\ell(p \circ \alpha_n | [a_n, t_n]) = \ell(p \circ \alpha_n | [t_n, b_n])$, and let $z_n = \alpha_n(t_n)$.

By compactness, there is a cluster point, z, of the sequence

z_n. It is clear that $\rho(x,z) = \rho(z,y) = 1/2 \, \rho(x,y)$.

   We are now ready to prove our theorem.

   Theorem: If T *is any involution on a Peano continuum* X

with nowhere dense fixed point set, then the T-*symmetric closed*

subsets of X *form a Hilbert sub-cube* S *of* 2^X *with Property Z,*

and the induced involution $\overline{T}$ *on the Hilbert cube* $2^X/\mathfrak{S}$ *is con-*

jugate to the standard reflection σ_∞ *of* $\prod_i J_i$.

   Proof: By Lemma 1, $\mathfrak{S}$ is a Hilbert sub-cube of the Hilbert

cube, $\tilde{\mathfrak{S}}$, and by Lemmas 2 and 3, $\mathfrak{S}$ has Property Z in 2^X. Thus,

by a theorem of Chapman ([4]; Property Z is the proper analogue

of cellularity, here.) $2^X/\tilde{\mathfrak{S}}$ is a Hilbert cube. By Lemma 4,

there is an invariant convex metric ρ on X, so using the Haus-

dorff metric on 2^X induced by ρ, the expansion homotopy $A \to N_t(A)$

is a continuous, equivariant contraction of 2^X to the point X.

This induces an equivariant contraction of $2^X/\tilde{\mathfrak{S}}$ to $\mathfrak{S}$ satisfying

the hypotheses of Wong's theorem [14], in that if d is the metric

induced on $2^X/\tilde{\mathfrak{S}}$ from ρ, the d-balls centered at $\mathfrak{S}$ are contracted

in themselves (equivariantly, which is even stronger). Thus,

$\overline{T}$ is standard.

3. Questions and Conjectures

   The above proof goes through *verbatim* (even the citation of

Wong) if the word "involution" is replaced by the phrase "periodic

mapping of prime period," and if we replace σ_∞ by a standard

model. My favorite is as follows: For any compact Lie group

G, let C(G) be the cone of G and let G act on C(G) by transla-

tion of the base (g([h,t]) = [gh,t], where [h,t] $\in$ C(G)

= G $\times$ [0,1]/G $\times$ {1}). Let $Q_G = \prod_{i=1}^{\infty} C(G)_i$, and let G act on Q_G

by the diagonal (simultaneous) action. Because C(G) is a con-

tractible polyhedron, it is a Q-factor [13], and an infinite

product of Q-factors is always a Hilbert cube [13]. (In fact,

R. Edwards [5] has shown all compact metric absolute retracts

to be Q-factors.) So we have a very canonical G action on a

Hilbert cube Q_G with single fixed point. (The action is free

off the infinite vertex.) This should be the standard.

   Question: What is the situation in respect to induced

G-actions on hyperspaces of Peano continua?

We note that in general this will be far more complex than
the involutive case, *e.g.*, if G is a compact, connected Lie group,
then the action of G on 2^G by left translation will not be semi-
free. In fact, if H is a proper, closed subgroup of G, then the
set of all members of 2^G which are fixed under the action of
each member of H is easily seen to be homeomorphic to the space
$2^{G/H}$ of closed subsets of the coset space and hence to the Hil-
bert cube. The initial aim should be to give conditions char-
acterizing G-translations among the G actions on Hilbert cubes.
Perhaps the first problem to be addressed is the following:

Conjecture: If G *is a compact, connected Lie group, is*
$2^G/G$ *a Hilbert cube?*

Finally, if G acts on the Peano continuum X, and if we
include X into 2^X as the singleton sets then G acts on 2^X by
the induced action and we can form $2^{(2^X)}$, etc. We obtain thus
a direct sequence $X \to 2^X \to 2^{(2^X)} \to \cdots$. If we give X a convex,
G-invariant metric, the inclusion into 2^X is an isometry and,
moreover, the Hausdorff metric is both G-invariant and convex.
Using the expansion homotopies $A \to N_t(A)$, we see that X is in-
cluded in 2^X as a Z-set. We can now (1) take the direct limit
$\varinjlim \{X \to 2^X\}$ of Hilbert cubes, obtaining Q_∞, which is homeo-
morphic to ℓ^2 with the bounded-weak topology (direct limit of the
closed bounded subsets of ℓ^2, each given the (analysts') weak
topology) [10]. *This model of* Q_∞ *has a G-action on it. How
does it compare with other G-actions on this space?*

We could, however, (2) take the metric direct limit of
$\{X \to 2^X \to \cdots\}$, *as the inclusions are isometries. What do we
get? Especially, if we complete the resulting space, what is
it? Is it, for example, homeomorphic to* ℓ^2? *And if so, what is
the induced G-action like in comparison with others?* (In this
connection, we might note that *free* actions of finite groups on

Hilbert spaces are all conjugate.   (See [15].)   These are not
free, however.)

     Finally, we restate a question of R. D. Anderson's:

     Question: Are all involutions of Q with unique fixed point
standard?   What about other compact groups (semi-free actions
with unique fixed points)?

References
1.   R. D. Anderson, *On topological infinite deficiency*, Mich.
     Math. J. 14 (1967), 365-383.
2.   Cz. Bessaga and A. Pelczynski, *Selected topics in infinite-*
     dimensional topology, Polish Academy of Sciences, Institute
     of Mathematics Monograph 58. Warsaw, 1975.
3.   R. H. Bing, *Partitioning a set*, Bull. Amer. Math. Soc. 55
     (1949), 1101-1110.
4.   T. A. Chapman, *On the structure of Hilbert cube manifolds*,
     Comp. Math. 24 (1972), 329-353.
5.   _____, *Lectures on Hilbert cube manifolds*, mimeographed
     notes. University of Kentucky.
6.   D. W. Curtis and R. M. Schori, 2^X *and C(X) are homeomorphic*
     to the Hilbert cube, Bull. Amer. Math. Soc. 78 (1972),
     402-406.
7.   _____, *Hyperspaces of polyhedra are Hilbert cubes*,
     preprint.
8.   _____, *Hyperspaces of Peano continua are Hilbert cubes*,
     preprint.
9.   J. Eells and N. H. Kuiper, *Homotopy negligible subsets of*
     infinite-dimensional manifolds, Comp. Math. 21 (1969),
     155-161.
10. R. E. Heisey, *Stability, open embeddings and classification*
     of bounded weak-* manifolds, preprint.
11. R. M. Schori and J. E. West, *The hyperspace of the closed*
     unit interval is a Hilbert cube, Trans. Amer. Math. Soc.
     213 (1975), 217-235.
12. _____, *Hyperspaces of graphs are Hilbert cubes*, Pac.
     J. Math. 53 (1974), 239-251.
13. J. E. West, *Infinite products which are Hilbert cubes*,
     Trans. Amer. Math. Soc. 150 (1970), 1-25.
14. R. Wong, *Periodic actions on the Hilbert cube*, Fund. Math.
     LXXXV (1974), 203-210.

15. _____, *Periodic actions on (I-D) normed linear spaces*,
    Fund. Math. LXXX (1973), 133-139.

Cornell University

Ithaca, New York 14853