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## WEAKLY COLLECTIONWISE HAUSDORFF SPACES

by

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**WEAKLY COLLECTIONWISE HAUSDORFF SPACES****Franklin D. Tall**<sup>1</sup>

The *Normal Moore Space Conjecture*, namely that normal Moore spaces are collectionwise normal and hence metrizable, is perhaps the outstanding open problem in set-theoretic topology. Partial results obtained by the author [T<sub>1</sub>] and W. Fleissner [F<sub>1</sub>] use non-elementary set-theoretic methods to achieve the consistency of e.g. normal Moore spaces being collectionwise Hausdorff. For a long time it was open whether the generalized continuum hypothesis (GCH) sufficed to achieve that result, but K. Devlin [D] has just proved that it does not. We shall show however that GCH does yield a weak variant.

*Definition.* Let  $\lambda$  be a cardinal. A space is (weakly)  $\lambda$ -collectionwise Hausdorff if for every closed discrete subspace of cardinality  $\lambda$ , there exist mutually disjoint open sets about ( $\lambda$  of) the points of the subspace. A space is (weakly) collectionwise Hausdorff if it is (weakly)  $\lambda$ -collectionwise Hausdorff for every  $\lambda$ .

*Theorem 1.* GCH implies every normal Moore space is weakly collectionwise Hausdorff.

*Corollary 2.* It is consistent with the usual axioms of set theory that every normal Moore space is weakly collectionwise Hausdorff, but that there is a normal Moore space that is not collectionwise Hausdorff.

As will be seen in the proof of Theorem 1, the failure of weak  $\lambda$ -collectionwise Hausdorffness functions as a weakening of the  $\lambda$ -chain condition (every collection of disjoint open sets

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has cardinality  $< \lambda$ ) which is yet sufficient for some of the purposes for which that condition is used. Indeed, our results and proofs are best viewed as generalizations of Jones [J] and Šapiroviskiĭ [Š].

Recall the *character* of a space is the supremum of minimal cardinalities of local neighborhood bases for points in the space. Fleissner actually proved that *Gödel's Axiom of Constructibility* implies every normal space of character  $\leq 2^{\aleph_0}$  is collectionwise Hausdorff. (Moore spaces have character  $\leq \aleph_0$ ). We shall prove

*Theorem 3. GCH implies every normal space of character  $\leq 2^{\aleph_0}$  is weakly collectionwise Hausdorff.*

The proof divides into a large number of cases, depending on what kind of cardinal  $\lambda$  is. We state the specifics in the following theorem from which Theorem 3 follows immediately.

*Theorem 4. a) If  $\lambda \leq \aleph_0$  and  $X$  is regular or normal,  $X$  is  $\lambda$ -collectionwise Hausdorff.*

*b) If  $\lambda = \kappa^+$  and  $2^\kappa < 2^\lambda$ , and if  $X$  is normal and has character  $\leq 2^\kappa$ , then  $X$  is weakly  $\lambda$ -collectionwise Hausdorff.*

*c) If  $\lambda$  is a regular limit cardinal such that  $2^\delta < 2^\lambda$ , and if  $X$  is normal and has character  $\leq 2^\delta$ , then  $X$  is weakly  $\lambda$ -collectionwise Hausdorff.*

*d) If  $\lambda$  is a singular cardinal with countable cofinality and  $X$  is normal and weakly  $\kappa$ -collectionwise Hausdorff for all  $\kappa < \lambda$  then  $X$  is weakly  $\lambda$ -collectionwise Hausdorff.*

*e) If  $\lambda$  is a singular cardinal with uncountable cofinality and for every  $\kappa < \lambda$ ,  $2^\kappa = \kappa^+$ , then if  $X$  is normal, has character less than  $\lambda$ , and is weakly  $\kappa$ -collectionwise Hausdorff for all  $\kappa < \lambda$ , then  $X$  is weakly  $\lambda$ -collectionwise Hausdorff.*

*The proof of a) is easy and well-known so is left to the*

reader.

*Proof of b).* Assume on the contrary that  $X$  is a normal space of character  $\leq 2^K$  and that  $Y$  is a closed discrete subspace of  $X$  of cardinality  $\lambda$  such that "separated" subsets of  $Y$  have cardinality  $\leq \kappa$ . Let  $Y = \{y_\alpha\}_{\alpha < \lambda}$ . Fix for each  $\alpha$  a neighbourhood base  $\mathcal{B}_\alpha$  for  $y_\alpha$ ,  $|\mathcal{B}_\alpha| \leq 2^K$ . By normality, for each  $Z \subseteq Y$  there is an open  $U_Z \supseteq Z$ ,  $\overline{U_Z} \cap (Y - Z) = \emptyset$ . For each  $Z$  pick a maximal disjoint collection  $\mathcal{S}_Z$  of elements of  $\mathcal{B}_\alpha$ 's,  $\alpha \in Z$ . Then  $Z_1 \neq Z_2$  implies  $\mathcal{S}_{Z_1} \neq \mathcal{S}_{Z_2}$  so the map  $Z \rightarrow \mathcal{S}_Z$  establishes that  $2^\lambda \leq (2^K \cdot \lambda)^K$  and hence  $2^\lambda = 2^K$ .

*The proof of case c)* is like that of b) and is therefore omitted. Recall that  $v^\lambda = \sum_{\kappa < \lambda} v^\kappa$ .

*To prove d),* partition  $\lambda$  into countably many pieces of smaller cardinality and use normality to separate them.

*The proof of e)* is a minor modification of Fleissner's proof  $[F_1]$  of the corresponding result for collectionwise Hausdorffness. We indicate the necessary change. Fleissner divides his closed discrete subspace  $Y$  into cofinality of  $\lambda$  disjoint equivalence classes, each of cardinality less than  $\lambda$ . By hypothesis each equivalence class is separated.  $Y$  is also divided into countably many disjoint pieces  $\{Y_i\}_{i < \omega}$ . These can be separated from each other by normality. Fleissner then proves that the traces of the equivalence classes on any  $Y_i$  can be separated from each other. By intersecting all these separations suitably, a separation of  $Y$  is obtained. We proceed in the same way save that given an equivalence class of cardinality  $\kappa$ , we separate  $\kappa$  of the points and "throw away" the rest. Since the equivalence classes cover  $Y$ , when we are through we have separated  $\lambda$  points.

Thanks to Fleissner  $[F_3]$  one expects cardinality arguments concerning closed discrete subspaces to work with minor changes

when normality is replaced by countable paracompactness. We do not as yet have the best possible results, but by modifying Fleissner's arguments we get

*Theorem 5. Assume  $(\lambda^\lambda)^{\aleph_0} = \lambda$ . Then every countably paracompact space of character  $\leq 2^\lambda$  is weakly  $\lambda$ -collectionwise Hausdorff.*

*Proof.* Let  $Y$  and  $\mathcal{B}_\alpha$  be as in the proof of Theorem 4. Let  $\{\mathbf{S}_\alpha: \alpha < \lambda\}$  index the countable sequences of disjoint collections of elements of the  $\mathcal{B}_\alpha$ 's. This can be done since  $((\lambda \cdot 2^\lambda)^\lambda)^{\aleph_0} = (\lambda^\lambda)^{\aleph_0} = \lambda$ . Let  $\mathbf{S}_\alpha = \{\mathcal{S}_{\alpha,n}: 0 < n < \omega\}$ . Partition  $Y$  into  $\{Y_n\}_{n < \omega}$  as follows. If for each  $n > 0$   $y_\alpha \in \overline{\mathcal{S}_{\alpha,n}}$ , then  $y_\alpha \in Y_0$ . Otherwise  $y_\alpha \in Y_n$ , where  $n$  is least such that  $y_\alpha \notin \overline{\mathcal{S}_{\alpha,n}}$ . Let  $V_n = X - (Y - Y_n)$ . If  $X$  is countably paracompact, there is a locally finite open refinement  $\mathcal{W}$  of  $\{V_n\}_{n < \omega}$ . Without loss of generality assume  $\mathcal{W} = \{W_n\}_{n < \omega}$  and  $W_n \subseteq V_n$ . For each  $n > 0$  pick a maximal disjoint collection  $\mathcal{S}_n$  of members of  $\mathcal{B}_\alpha$ 's included in  $W_n$ . Note by maximality that  $\overline{\mathcal{S}_n} \supseteq W_n \cap Y = Y_n$ . There is an  $\alpha$  such that  $\{\mathcal{S}_n: 0 < n < \omega\} = \mathbf{S}_\alpha$ . Consider  $y_\alpha$ . If  $y_\alpha \in Y_0$ , then  $\{\overline{W}_n\}_{n < \omega}$  is not locally finite, so neither is  $\mathcal{W}$ . If  $y_\alpha \in Y_n$ ,  $n > 0$ , then  $y_\alpha \notin \overline{\mathcal{S}_n}$ , contradiction.

Eric van Douwen has disposed of  $\lambda$ 's with countable cofinality and has kindly permitted us to include his result:

*Theorem 6. If  $X$  is countably paracompact, it is weakly  $\aleph_0$ -collectionwise Hausdorff. If  $\lambda$  is a cardinal of countable cofinality and  $X$  is also weakly  $\kappa$ -collectionwise Hausdorff for all  $\kappa < \lambda$ , then  $X$  is weakly  $\lambda$ -collectionwise Hausdorff.*

*Proof.* Let  $X$  be countably paracompact. Let  $\lambda$  be an infinite cardinal of countable cofinality. If  $\lambda$  is uncountable, assume  $X$  is weakly  $\kappa$ -collectionwise Hausdorff for all  $\kappa < \lambda$ . Let  $Y$  be a closed discrete subspace of  $X$  of cardinality  $\lambda$ . Let

$\{Y_n\}_{n < \omega}$  be a partition of  $Y$  such that

- 1) if  $\lambda = \aleph_0$ ,  $|Y_n| = 1$  for all  $n$ ,
- 2) if  $\lambda > \aleph_0$ ,  $|Y_n|$  is a regular cardinal and  $|Y_n| < |Y_{n+1}|$  for all  $n$ .

By hypothesis, for each  $n$  there is  $Z_n \subseteq Y_n$ ,  $|Z_n| = |Y_n|$ , and open sets  $\{V(x) : x \in Z_n\}$ , such that  $x \in V(x)$  and for distinct  $x, y \in Z_n$ ,  $V(x) \cap V(y) = \emptyset$ . Let  $Z = \bigcup_{n < \omega} Z_n$ . Let  $\{U_n : n < \omega\}$  be a precise locally finite open refinement of  $\{(X - (Z - Z_n) : n < \omega)\}$ . For each  $x \in Z_n$  choose a neighbourhood  $N(x) \subseteq V(x) \cap U_n$  which intersects only finitely many  $U_k$ 's. For  $n, k < \omega$  define

$$Z_{n,k} = \{x \in Z_n : \text{for all } m \geq k, N(x) \cap U_m = \emptyset\}.$$

Since  $|Z_n|$  is regular, there is an  $f \in {}^\omega \omega$  such that  $|Z_{n,f(n)}| = |Z_n|$ . Define  $g \in {}^\omega \omega$  by  $g(0) = 0$ ,  $g(n+1) = f(g(n))$ . Let  $F_n = Z_{n,f(n)}$ . Let  $F = \bigcup_{n < \omega} F_{g(n)}$ . Then  $|F| = \lambda$  and  $N(x) \cap N(y) = \emptyset$  for any two distinct  $x, y \in F$ .

*Corollary 7. GCH implies every countably paracompact space of character  $\leq 2^{\aleph_0}$  is weakly  $\lambda$ -collectionwise Hausdorff for every regular  $\lambda$  and every singular  $\lambda$  with countable cofinality.*

We conjecture that the Corollary is true for all cardinals. It is possible to formulate versions of our results without recourse to cardinal arithmetic hypotheses but by no means without cardinal arithmetic. We give an example and leave its proof and generalizations to the reader.

*Theorem 8. Suppose  $X$  is normal or countably paracompact and has character  $\leq 2^{\aleph_0}$ . Then if  $Y$  is a closed discrete subspace of  $X$  of cardinality  $2^{\aleph_0}$ , there exist mutually disjoint open sets about  $\aleph_1$  points of  $Y$ .*

Applications of collectionwise Hausdorffness are rare; it is therefore to be expected that applications of the weak variant are rarer. We do have one however, and another one appears

in [GW]. We state the simplest case, which strengthens results of Šapirovič [Š].

*Definition.* A space satisfies the *discrete countable chain condition* [W] if every discrete collection of open sets is countable. A space is  $\aleph_1$ -compact if every closed discrete subspace is countable.

*Theorem 9.*  $2^{\aleph_0} < 2^{\aleph_1}$  implies every normal space of character  $\leq 2^{\aleph_0}$  satisfying the discrete countable chain condition is  $\aleph_1$ -compact.

*Proof.* If such a space had an uncountable closed discrete subspace, by Theorem 4b) there would exist pairwise disjoint open sets about uncountably many of the points. By normality these open sets could be shrunk to a discrete collection.

Aside from the singular cardinal of uncountable cofinality case for countable paracompactness which we cannot settle, there are several other problems. Can normality be replaced by countable paracompactness in the consistency results alluded to on page 1? Can "weakly collectionwise Hausdorff" be improved to "weakly collectionwise normal"?

We next turn our attention to examples.

*Example 1.* Bing's Example H  $[B_1]$  is perfectly normal (hence countably paracompact), has character  $2^{\aleph_1}$  and is not weakly  $\aleph_1$ -collectionwise Hausdorff.

*Example 2.* Assuming Martin's Axiom plus  $2^{\aleph_0} > \aleph_1$ , there is a separable normal non-metrizable Moore space  $[T_1]$ . Such a space is normal, countably paracompact, and not weakly  $\aleph_1$ -collectionwise Hausdorff.

*Example 3.*  $X$  is a special Aronszajn tree with the induced tree topology. In  $[F_2]$  or  $[R]$  the space is defined, shown to be

a non-collectionwise Hausdorff Moore space, and proved to be normal or not, depending upon what set-theoretic axioms are assumed. Devlin [D] has recently shown GCH to be consistent with the normality of some  $X$ . We shall prove outright that  $X$  is weakly collectionwise Hausdorff. Indeed let  $T$  be any tree of height  $\omega_1$  with countable levels. For  $t \in T$  let  $h(t)$  be the ordertype of the set of predecessors of  $t$ . Let  $\{t_\alpha\}_{\alpha < \omega_1}$  be distinct elements of  $T$ . Define  $\{s_\beta\}_{\beta < \omega_1}$  by letting  $s_\beta$  be the first  $t_\alpha$  such that  $h(t_\alpha) > \sup\{h(s_\gamma) : \gamma < \beta\}$ . Define  $g(\beta) = \sup\{h(s_\gamma) : \gamma < \beta\}$ . Let  $r_\beta$  be that predecessor of  $s_\beta$  of height  $g(\beta)$ . Let  $U_\beta = \{t \in T : s_\beta \leq t < r_\beta\}$ . Then  $\{U_\beta\}_{\beta < \omega_1}$  is a disjoint collection of open sets about  $\{s_\beta\}_{\beta < \omega_1}$ . Thus  $T$  is weakly collectionwise Hausdorff.

Alster and Pol [AP] prove that collectionwise Hausdorff locally separable Moore spaces are metrizable. Since  $\omega_1$ -trees are locally separable, weakly collectionwise Hausdorff will not suffice.

Fleissner--to whom I am grateful for many helpful comments--points out that Example 3 can be generalized. Let  $\lambda$  be a cardinal; for every  $\alpha < \lambda$  of countable cofinality, let  $s_\alpha$  be a sequence cofinal in  $\alpha$ . Let the points of  $X$  be the  $s_\alpha$ 's and the initial segments of the  $s_\alpha$ 's. Let  $X$  be endowed with the tree topology. Then for all  $\kappa \leq \lambda$ ,  $X$  is weakly  $\kappa$ -collectionwise Hausdorff but not  $\kappa$ -collectionwise Hausdorff.

As noted in  $[B_2]$  and  $[T_2]$ , the question of whether normal first countable spaces are collectionwise Hausdorff has an attractive set-theoretic translation. The same technique translates questions involving weakly collectionwise Hausdorffness. We shall translate some of the results we have proved here; that these are in fact translations will be evident to the reader of  $[T_2]$ .

*Definition.* Let  ${}^{\omega_1}\omega$  be the set of functions from  $\omega_1$  to  $\omega$ . Let  $G \subseteq {}^{\omega_1}\omega$ . Let  $Y \subseteq \omega_1$ .  $G$  is doubly superior on  $Y$  if  $(\forall f \in {}^{\omega_1}\omega)(\exists g \in G)(\exists \alpha_0, \alpha_1 \in Y)(g(\alpha_0) > f(\alpha_0), g(\alpha_1) > f(\alpha_1))$ .  $G$  is doubly superior if  $Y = \omega_1$ . Let  $p \subseteq \omega_1 \times 2$  be a function,  $\text{domain } p \subseteq \omega_1$ ,  $\text{range } p \subseteq 2$ .  $p$  splits  $G$  if  $(\forall f \in {}^{\omega_1}\omega)(\exists g \in G)(\exists \alpha_0, \alpha_1 \in \text{domain } p, p(\alpha_0) \neq p(\alpha_1))(g(\alpha_0) > f(\alpha_0), g(\alpha_1) > f(\alpha_1))$ .  $G$  splits if there is a  $p$  which splits it.

*Theorem 10.* a) if  $2^{\aleph_0} < 2^{\aleph_1}$  and  $G$  is doubly superior on every uncountable set, then  $G$  splits,

b) there is a  $G$  which is doubly superior but is not doubly superior on every uncountable set, and such that it is independent of the usual axioms of set theory with or without GCH whether or not  $G$  splits,

c) Martin's Axiom plus  $2^{\aleph_0} > \aleph_1$  implies there is a  $G$  which is doubly superior on every uncountable set but does not split.

*Added in proof.* S. Shelah has pointed out that better results may be obtained at singular cardinals by more careful calculation of cardinalities, and has kindly suggested I include his improvements.

*Theorem 11.* Suppose  $\lambda$  is singular and  $\chi^\lambda < 2^\lambda$ . Then if  $X$  is a normal space of character  $\leq \chi$  which is weakly  $\kappa$ -collectionwise Hausdorff for each  $\kappa < \lambda$ ,  $X$  is weakly  $\lambda$ -collectionwise Hausdorff.

*Proof.* Let  $Z, U_Z$  be as in the proof of Theorem 4. Let  $\lambda = \Sigma\{\lambda_\beta : \lambda < \text{cf}(\lambda)\}$ . Take  $\lambda_0$  disjoint basic open sets included in  $U_Z$  about  $\alpha$ 's in  $\lambda_0 \cap Z$ . Enlarge the collection to get  $\lambda_1$  disjoint basic open sets about  $\alpha$ 's in  $\lambda_1 \cap Z$ . Keep enlarging the disjoint collection, skipping some  $\lambda_\beta$ 's if necessary.

Either we get permanently stuck at some  $\beta < \text{cf}(\lambda)$  or we are done. Assume the former happens for all  $Z$ . There are  $\chi^\lambda$  such finished collections  $\mathfrak{S}_Z$ . There exist  $2^\lambda$  subsets of  $\lambda$  such that any two have a difference of power  $\lambda$ . On this family, the map  $Z \rightarrow \mathfrak{S}_Z$  is one-one, assuming weak  $\kappa$ -collectionwise Hausdorffness for all  $\kappa < \lambda$ .

*Corollary 12.* Suppose for every  $\lambda$ ,  $2^\lambda < 2^{\lambda^+}$ . Then every normal space of character  $\leq 2^{\aleph_0}$  is weakly collectionwise Hausdorff.

*Proof.* For  $\lambda$  a limit cardinal, the cofinality of  $2^\lambda$  is  $\lambda$ , hence by König's Lemma,  $2^\lambda < 2^\lambda$ .

*Theorem 13.* Suppose  $(\chi^\lambda)^{\aleph_0} = \lambda$ . If  $X$  is a countably paracompact space of character  $\leq \chi$  which is weakly  $\kappa$ -collectionwise Hausdorff for each  $\kappa < \lambda$ , then  $X$  is weakly  $\lambda$ -collectionwise Hausdorff.

*Proof.* We modify the proof of Theorem 5 using the ideas of the proof of Theorem 11. By the cardinality hypothesis we may enumerate in a sequence of type  $\lambda$  all countable sequences of disjoint collections constructed as in the proof of Theorem 11, so that each such countable sequence appears  $\lambda$  times. Choose  $Y_n, V_n, W_n$  as in the proof of Theorem 5, and then construct  $\mathfrak{S}_n$  as in the proof of Theorem 11, with  $Y_n$  and  $W_n$  playing the role of  $Z$  and  $U_Z$ . Since  $\text{cf}(\lambda) > \aleph_0$ , there is a  $\kappa < \lambda$  such that  $\kappa > \sup\{\beta: y_\beta \in \mathfrak{S}_n \text{ for some } n\}$ . For no  $n$  does  $Y_n - \overline{U \mathfrak{S}_n}$  include a separated subset of power  $\kappa$ . By induction hypothesis then  $|Y_n - \overline{U \mathfrak{S}_n}| < \kappa$  for all  $n$ . But proceeding as in the proof of Theorem 5, we get  $\lambda$   $\alpha$ 's in  $Y_n - \overline{U \mathfrak{S}_n}$ , contradiction.

*Corollary 14.* GCH implies every countably paracompact space of character  $\leq 2^{\aleph_0}$  is weakly collectionwise Hausdorff.

*Added in Proof:* Devlin has withdrawn his claim that GCH is consistent with the existence of a normal Aronszajn tree.

Instead, he now claims that  $2^0 \leq 2^1$  implies that an Aronszajn tree is normal iff and only if it is Souslin. However, Shelah claims that GCH is consistent with the existence of a normal, nonmetrizable Moore space. His space is a modification of a special Aronszajn tree.

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