WEAKLY COLLECTIONWISE
HAUSDORFF SPACES

by

Franklin D. Tall

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The *Normal Moore Space Conjecture*, namely that normal Moore spaces are collectionwise normal and hence metrizable, is perhaps the outstanding open problem in set-theoretic topology. Partial results obtained by the author $[T_1]$ and W. Fleissner $[F_1]$ use non-elementary set-theoretic methods to achieve the consistency of e.g. normal Moore spaces being collectionwise Hausdorff. For a long time it was open whether the generalized continuum hypothesis (GCH) sufficed to achieve that result, but K. Devlin [D] has just proved that it does not. We shall show however that GCH does yield a weak variant.

Definition. Let λ be a cardinal. A space is *(weakly)* λ-*collectionwise Hausdorff* if for every closed discrete subspace of cardinality λ, there exist mutually disjoint open sets about (λ of) the points of the subspace. A space is *(weakly) collectionwise Hausdorff* if it is (weakly) λ-collectionwise Hausdorff for every λ.

Theorem 1. GCH *implies every normal Moore space is weakly collectionwise Hausdorff.*

Corollary 2. It is consistent with the usual axioms of set theory that every normal Moore space is weakly collectionwise Hausdorff, but that there is a normal Moore space that is not collectionwise Hausdorff.

As will be seen in the proof of Theorem 1, the failure of weak λ-collectionwise Hausdorffness functions as a weakening of the λ-*chain condition* (every collection of disjoint open sets

has cardinality $<\lambda$) which is yet sufficient for some of the
purposes for which that condition is used. Indeed, our results
and proofs are best viewed as generalizations of Jones [J] and
Šapirovskiĭ [Š].

Recall the *character* of a space is the supremum of minimal
cardinalities of local neighborhood bases for points in the
space. Fleissner actually proved that *Gödel's Axiom of Con-
structibility* implies every normal space of character $\leq 2^{\aleph_0}$ is
collectionwise Hausdorff. (Moore spaces have character $\leq \aleph_0$).
We shall prove

*Theorem 3. GCH implies every normal space of character
$\leq 2^{\aleph_0}$ is weakly collectionwise Hausdorff.*

The proof divides into a large number of cases, depending
on what kind of cardinal λ is. We state the specifics in the
following theorem from which Theorem 3 follows immediately.

*Theorem 4. a) If $\lambda \leq \aleph_0$ and X is regular or normal, X is
λ-collectionwise Hausdorff.*

*b) If $\lambda = \kappa^+$ and $2^\kappa < 2^\lambda$, and if X is normal
and has character $\leq 2^\kappa$, then X is weakly λ-collectionwise Haus-
dorff.*

*c) If λ is a regular limit cardinal such that
$2^\delta < 2^\lambda$, and if X is normal and has character $\leq 2^\delta$, then X is
weakly λ-collectionwise Hausdorff.*

*d) If λ is a singular cardinal with countable
cofinality and X is normal and weakly κ-collectionwise Hausdorff
for all $\kappa < \lambda$ then X is weakly λ-collectionwise Hausdorff.*

*e) If λ is a singular cardinal with uncountable
cofinality and for every $\kappa < \lambda$, $2^\kappa = \kappa^+$, then if X is normal,
has character less than λ, and is weakly κ-collectionwise Haus-
dorff for all $\kappa < \lambda$, then X is weakly λ-collectionwise Hausdorff.*

The proof of a) is easy and well-known so is left to the

reader.

Proof of b). Assume on the contrary that X is a normal
space of character $\leq 2^{\kappa}$ and that Y is a closed discrete subspace
of X of cardinality λ such that "separated" subsets of Y have
cardinality $\leq \kappa$. Let $Y = \{y_{\alpha}\}_{\alpha < \lambda}$. Fix for each α a neighbour-
hood base $\mathcal{B}_{\alpha}$ for y_{α}, $|\mathcal{B}_{\alpha}| \leq 2^{\kappa}$. By normality, for each $Z \subseteq Y$
there is an open $U_Z \supseteq Z$, $\bar{U}_Z \cap (Y - Z) = \emptyset$. For each Z pick a
maximal disjoint collection $\mathcal{S}_Z$ of elements of $\mathcal{B}_{\alpha}$'s, $\alpha \in Z$.
Then $Z_1 \neq Z_2$ implies $\mathcal{S}_{Z_1} \neq \mathcal{S}_{Z_2}$ so the map $Z \to \mathcal{S}_Z$ establishes
that $2^{\lambda} \leq (2^{\kappa} \cdot \lambda)^{\kappa}$ and hence $2^{\lambda} = 2^{\kappa}$.

The proof of case c) is like that of b) and is therefore
omitted. Recall that $v^{\delta} = \sum_{\kappa < \lambda} v^{\kappa}$.

To prove d), partition λ into countably many pieces of
smaller cardinality and use normality to separate them.

The proof of e) is a minor modification of Fleissner's
proof [F_1] of the corresponding result for collectionwise
Hausdorffness. We indicate the necessary change. Fleissner
divides his closed discrete subspace Y into cofinality of λ
disjoint equivalence classes, each of cardinality less than λ.
By hypothesis each equivalence class is separated. Y is also
divided into countably many disjoint pieces $\{Y_i\}_{i < \omega}$. These
can be separated from each other by normality. Fleissner then
proves that the traces of the equivalence classes on any Y_i can
be separated from each other. By intersecting all these separa-
tions suitably, a separation of Y is obtained. We proceed in
the same way save that given an equivalence class of cardinality
κ, we separate κ of the points and "throw away" the rest. Since
the equivalence classes cover Y, when we are through we have
separated λ points.

Thanks to Fleissner [F_3] one expects cardinality arguments
concerning closed discrete subspaces to work with minor changes

when normality is replaced by countable paracompactness. We

do not as yet have the best possible results, but by modifying

Fleissner's arguments we get

   Theorem 5. Assume $(\lambda^{\delta})^{\aleph_0} = \lambda$*. Then every countably*

paracompact space of character $\leq 2^{\delta}$ *is weakly* λ*-collectionwise*

Hausdorff.

   Proof. Let Y and $\mathcal{B}_{\alpha}$ be as in the proof of Theorem 4. Let

$\{\mathbf{S}_{\alpha}: \alpha < \lambda\}$ index the countable sequences of disjoint collec-

tions of elements of the $\mathcal{B}_{\alpha}$'s. This can be done since

$((\lambda \cdot 2^{\delta})^{\delta})^{\aleph_0} = (\lambda^{\delta})^{\aleph_0} = \lambda$. Let $\mathbf{S}_{\alpha} = \{\mathcal{S}_{\alpha,n}: 0 < n < \omega\}$.

Partition Y into $\{Y_n\}_{n<\omega}$ as follows. If for each $n > 0$

$y_{\alpha} \in \overline{\cup \mathcal{S}}_{\alpha,n}$, then $y_{\alpha} \in Y_0$. Otherwise $y_{\alpha} \in Y_n$, where n is least

such that $y_{\alpha} \notin \overline{\cup \mathcal{S}}_{\alpha,n}$. Let $V_n = X - (Y - Y_n)$. If X is counta-

bly paracompact, there is a locally finite open refinement $\mathcal{W}$

of $\{V_n\}_{n<\omega}$. Without loss of generality assume $\mathcal{W} = \{W_n\}_{n<\omega}$ and

$W_n \subseteq V_n$. For each $n > 0$ pick a maximal disjoint collection $\mathcal{S}_n$

of members of $\mathcal{B}_{\alpha}$'s included in W_n. Note by maximality that

$\overline{\cup \mathcal{S}}_n \supseteq W_n \cap Y = Y_n$. There is an α such that $\{\mathcal{S}_n: 0 < n < \omega\} = \mathbf{S}_{\alpha}$.

Consider y_{α}. If $y_{\alpha} \in Y_0$, then $\{\overline{W}_n\}_{n<\omega}$ is not locally finite, so

neither is $\mathcal{W}$. If $y_{\alpha} \in Y_n$, $n > 0$, then $y_{\alpha} \notin \overline{\cup \mathcal{S}}_n$, contradiction.

   Eric van Douwen has disposed of λ's with countable co-

finality and has kindly permitted us to include his result:

   Theorem 6. If X *is countably paracompact, it is weakly*

$\aleph_0$*-collectionwise Hausdorff. If* λ *is a cardinal of countable*

cofinality and X *is also weakly* κ*-collectionwise Hausdorff for*

all $\kappa < \lambda$*, then* X *is weakly* λ*-collectionwise Hausdorff.*

   Proof. Let X be countably paracompact. Let λ be an in-

finite cardinal of countable cofinality. If λ is uncountable,

assume X is weakly κ-collectionwise Hausdorff for all $\kappa < \lambda$.

Let Y be a closed discrete subspace of X of cardinality λ. Let

$\{Y_n\}_{n<\omega}$ be a partition of Y such that

1) if $\lambda = \aleph_0$, $|Y_n| = 1$ for all n,

2) if $\lambda > \aleph_0$, $|Y_n|$ is a regular cardinal and $|Y_n| < |Y_{n+1}|$

   for all n.

By hypothesis, for each n there is $Z_n \subseteq Y_n$, $|Z_n| = |Y_n|$,

and open sets $\{V(x): x \in Z_n\}$, such that $x \in V(x)$ and for dis-

tinct $x,y \in Z_n$, $V(x) \cap V(y) = \emptyset$. Let $Z = \bigcup_{n<\omega} Z_n$. Let $\{U_n: n < \omega\}$

be a precise locally finite open refinement of $\{(X - (Z - Z_n):$

$n < \omega\}$. For each $x \in Z_n$ choose a neighbourhood $N(x) \subseteq V(x) \cap U_n$

which intersects only finitely many U_k's. For $n,k < \omega$ define

$$Z_{n,k} = \{x \in Z_n: \text{ for all } m \geq k, N(x) \cap U_m = \emptyset\}.$$

Since $|Z_n|$ is regular, there is an $f \in {}^\omega\omega$ such that $|Z_{n,f(n)}| = $

$|Z_n|$. Define $g \in {}^\omega\omega$ by $g(0) = 0$, $g(n + 1) = f(g(n))$. Let

$F_n = Z_{n,f(n)}$. Let $F = \bigcup_{n<\omega} F_{g(n)}$. Then $|F| = \lambda$ and $N(x) \cap N(y) = \emptyset$

for any two distinct $x,y \in F$.

Corollary 7. GCH implies every countably paracompact space

of character $\leq 2^{\aleph_0}$ is weakly λ-collectionwise Hausdorff for

every regular λ and every singular λ with countable cofinality.

We conjecture that the Corollary is true for all cardinals.

It is possible to formulate versions of our results without re-

course to cardinal arithmetic hypotheses but by no means without

cardinal arithmetic. We give an example and leave its proof and

generalizations to the reader.

Theorem 8. Suppose X is normal or countably paracompact and

has character $\leq 2^{\aleph_0}$. Then if Y is a closed discrete subspace of

X of cardinality $2^{\aleph_0}$, there exist mutually disjoint open sets

about $\aleph_1$ points of Y.

Applications of collectionwise Hausdorffness are rare; it

is therefore to be expected that applications of the weak vari-

ant are rarer. We do have one however, and another one appears

in [GW]. We state the simplest case, which strengthens results

of Šapirovskiĭ [Š].

   Definition. A space satisfies the *discrete countable chain*

condition [W] if every discrete collection of open sets is

countable. A space is $\aleph_1$-*compact* if every closed discrete sub-

space is countable.

   Theorem 9. $2^{\aleph_0} < 2^{\aleph_1}$ *implies every normal space of*

character $\leq 2^{\aleph_0}$ *satisfying the discrete countable chain condi-*

tion is $\aleph_1$-*compact.*

   Proof. If such a space had an uncountable closed discrete

subspace, by Theorem 4b) there would exist pairwise disjoint

open sets about uncountably many of the points. By normality

these open sets could be shrunk to a discrete collection.

   Aside from the singular cardinal of uncountable cofinality

case for countable paracompactness which we cannot settle, there

are several other problems. Can normality be replaced by

countable paracompactness in the consistency results alluded to

on page 1? Can "weakly collectionwise Hausdorff" be improved to

"weakly collectionwise normal"?

   We next turn our attention to examples.

   Example 1. Bing's Example H [B_1] is perfectly normal (hence

countably paracompact), has character $2^{\aleph_1}$ and is not weakly

$\aleph_1$-collectionwise Hausdorff.

   Example 2. Assuming Martin's Axiom plus $2^{\aleph_0} > \aleph_1$, there

is a separable normal non-metrizable Moore space [T_1]. Such a

space is normal, countably paracompact, and not weakly $\aleph_1$-

collectionwise Hausdorff.

   Example 3. X is a special Aronszajn tree with the induced

tree topology. In [F_2] or [R] the space is defined, shown to be

a non-collectionwise Hausdorff Moore space, and proved to be
normal or not, depending upon what set-theoretic axioms are
assumed. Devlin [D] has recently shown GCH to be consistent
with the normality of some X. We shall prove outright that X
is weakly collectionwise Hausdorff. Indeed let T be any tree of
height ω_1 with countable levels. For $t \in T$ let $h(t)$ be the
ordertype of the set of predecessors of t. Let $\{t_\alpha\}_{\alpha<\omega_1}$ be
distinct elements of T. Define $\{s_\beta\}_{\beta<\omega_1}$ by letting s_β be the
first t_α such that $h(t_\alpha) > \sup\{h(s_\gamma): \gamma < \beta\}$. Define $g(\beta) =$
$\sup\{h(s_\gamma): \gamma < \beta\}$. Let r_β be that predecessor of s_β of height
$g(\beta)$. Let $U_\beta = \{t \in T: s_\beta \geq t > r_\beta\}$. Then $\{U_\beta\}_{\beta<\omega_1}$ is a dis-
joint collection of open sets about $\{s_\beta\}_{\beta<\omega_1}$. Thus T is weakly
collectionwise Hausdorff.

   Alster and Pol [AP] prove that collectionwise Hausdorff
locally separable Moore spaces are metrizable. Since ω_1-trees
are locally separable, weakly collectionwise Hausdorff will not
suffice.

   Fleissner--to whom I am grateful for many helpful comments--
points out that Example 3 can be generalized. Let λ be a cardi-
nal; for every $\alpha < \lambda$ of countable cofinality, let s_α be a sequence
cofinal in α. Let the points of X be the s_α's and the initial
segments of the s_α's. Let X be endowed with the tree topology.
Then for all $\kappa \leq \lambda$, X is weakly κ-collectionwise Hausdorff but
not κ-collectionwise Hausdorff.

   As noted in $[B_2]$ and $[T_2]$, the question of whether normal
first countable spaces are collectionwise Hausdorff has an at-
tractive set-theoretic translation. The same technique trans-
lates questions involving weakly collectionwise Hausdorffness.
We shall translate some of the results we have proved here; that
these are in fact translations will be evident to the reader of
$[T_2]$.

Definition. Let $^{\omega_1}\omega$ be the set of functions from ω_1 to ω.
Let $G \subseteq {}^{\omega_1}\omega$. Let $Y \subseteq \omega_1$. G *is doubly superior on* Y if
$(\forall f \in {}^{\omega_1}\omega)(\exists g \in G)(\exists \alpha_0, \alpha_1 \in Y)(g(\alpha_0) > f(\alpha_0),\ g(\alpha_1) > f(\alpha_1))$.
G is *doubly superior* if $Y = \omega_1$. Let $p \subseteq \omega_1 \times 2$ be a function,
domain $p \subseteq \omega_1$, range $p \subseteq 2$. p *splits* G if
$(\forall f \in {}^{\omega_1}\omega)(\exists g \in G)(\exists \alpha_0, \alpha_1 \in$ domain $p,\ p(\alpha_0) \neq p(\alpha_1))(g(\alpha_0) >$
$f(\alpha_0),\ g(\alpha_1) > f(\alpha_1))$.
G *splits* if there is a p which splits it. ·

Theorem 10. a) *if* $2^{\aleph_0} < 2^{\aleph_1}$ *and* G *is doubly superior on
every uncountable set, then* G *splits,*

b) *there is a* G *which is doubly superior but
is not doubly superior on every uncountable set, and such that
it is independent of the usual axioms of set theory with or
without GCH whether or not* G *splits,*

c) *Martin's Axiom plus* $2^{\aleph_0} > \aleph_1$ *implies there
is a* G *which is doubly superior on every uncountable set but
does not split.*

Added in proof. S. Shelah has pointed out that better
results may be obtained at singular cardinals by more careful
calculation of cardinalitites, and has kindly suggested I include
his improvements.

Theorem 11. Suppose λ *is singular and* $\chi^\delta < 2^\lambda$. *Then if*
X *is a normal space of character* $\leq \chi$ *which is weakly* κ-*collection-
wise Hausdorff for each* $\kappa < \lambda$, X *is weakly* λ-*collectionwise
Hausdorff.*

Proof. Let Z, U_Z be as in the proof of Theorem 4. Let
$\lambda = \Sigma\{\lambda_\beta : \lambda < \text{cf}(\lambda)\}$. Take λ_0 disjoint basic open sets included
in U_Z about α's in $\lambda_0 \cap Z$. Enlarge the collection to get λ_1
disjoint basic open sets about α's in $\lambda_1 \cap Z$. Keep enlarging
the disjoint collection, skipping some λ_β's if necessary.

Either we get permanently stuck at some $\beta < \mathrm{cf}(\lambda)$ or we are done. Assume the former happens for all Z. There are χ^{λ} such finished collections $\mathfrak{S}_Z$. There exist 2^{λ} subsets of λ such that any two have a difference of power λ. On this family, the map $Z \rightarrow \mathfrak{S}_Z$ is one-one, assuming weak κ-collectionwise Hausdorffness for all $\kappa < \lambda$.

   Corollary 12. *Suppose for every* λ, $2^{\lambda} < 2^{\lambda^{+}}$. *Then every normal space of character* $\leq 2^{\aleph_0}$ *is weakly collectionwise Hausdorff.*

   Proof. For λ a limit cardinal, the cofinality of 2^{λ} is λ, hence by König's Lemma, $2^{\lambda} < 2^{\lambda}$.

   Theorem 13. *Suppose* $(\chi^{\lambda})^{\aleph_0} = \lambda$. *If* X *is a countably paracompact space of character* $\leq \chi$ *which is weakly* κ-*collectionwise Hausdorff for each* $\kappa < \lambda$, *then* X *is weakly* λ-*collectionwise Hausdorff.*

   Proof. We modify the proof of Theorem 5 using the ideas of the proof of Theorem 11. By the cardinality hypothesis we may enumerate in a sequence of type λ all countable sequences of disjoint collections constructed as in the proof of Theorem 11, so that each such countable sequence appears λ times. Choose Y_n, V_n, W_n as in the proof of Theorem 5, and then construct $\mathfrak{S}_n$ as in the proof of Theorem 11, with Y_n and W_n playing the role of Z and U_Z. Since $\mathrm{cf}(\lambda) > \aleph_0$, there is a $\kappa < \lambda$ such that $\kappa > \sup\{\beta: y_{\beta} \in \mathfrak{S}_n$ for some n$\}$. For no n does $Y_n - \overline{U \mathfrak{S}}_n$ include a separated subset of power κ. By induction hypothesis then $|Y_n - \overline{U \mathfrak{S}}_n| < \kappa$ for all n. But proceeding as in the proof of Theorem 5, we get λ α's in $Y_n - \overline{U \mathfrak{S}}_n$, contradiction.

   Corollary 14. GCH *implies every countably paracompact space of character* $\leq 2^{\aleph_0}$ *is weakly collectionwise Hausdorff.*

   Added in Proof: Devlin has withdrawn his claim that GCH is consistant with the existance of a normal Aronszajn tree.

Instead, he now claims that $2^0 \leq 2^1$ implies that an Aronszajn tree is normal iffand only if it is Souslin. However, Shelah claims that GCH is consistant with the existance of a normal, nonmetrizable Moore space. His space is a modification of a special Aronszajn tree.

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