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Research Announcement:
EXTENSIONS OF FIRST COUNTABLE AND OF COUNTABLE SPACES

by

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EXTENSIONS OF FIRST COUNTABLE AND OF COUNTABLE SPACES

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The results that are subject of my talk have been obtained jointly with Eric K. van Douwen. All spaces are regular and $c = 2^{\omega_0}$.

Part I. First countable and countable spaces all compactifications of which contain βN

Example 1. A first countable Lindelöf (even cosmic) space Δ *all compactifications of which contain* βN.

Example 2. A countable space Σ *with one non-isolated point all compactifications of which contain* βN.

Since βN has cardinality 2^c, uncountable tightness and is neither first countable nor scattered, the above examples in particular yield:

(1) A first countable Lindelöf space with no first countable compactification.

(2) A countable space all compactifications of which have cardinality 2^c and uncountable tightness.

(3) A scattered space with no scattered compactification.

Complicated examples of spaces satisfying one of the conditions (1)-(3) have been recently constructed (in answer to the questions of V. I. Ponomariev, A. V. Arhangel'skiĭ and R. Telgarsky) by Ul'janov [U], Efimov [E], Malyhin [M] and Nyikos [N].

Our examples, however, are simpler and have better properties. The space Δ is the union of the unit interval I and of a countable discrete space N and $\Sigma = \Delta/I$. Therefore Σ is a Frèchet space.

Part II. Separable extensions of first countable spaces

Fact. A space X can be embedded into a separable space if and only if the weight of X is $\leq c$.

Problem 1. (J. W. Ott, B. Fitzpatrick, G. M. Reed) *Can each first countable (resp. Moore) space of weight* $\leq c$ *be embedded into a separable first countable (resp. Moore) space?*

It turns out that the answer YES to the above Problem is consistent with but also independent of the ZFC axioms of set theory.

Theorem 1. Every first countable (resp. Moore) space of weight $\leq \omega_1$ *can be embedded into a separable first countable (resp. Moore) space.*

Theorem 1 shows that the Continuum Hypothesis implies a positive answer to the Problem 1. To show that also a negative answer is consistent, let us first consider the following set-theoretic condition:

$T(\lambda)$: For every family $\mathcal{A}$ of cardinality λ of subsets of a set X of cardinality λ there exists a separable metric ρ on X such that all $A \in \mathcal{A}$ are F_σ subsets of (X, ρ).

Proposition 1. The negation of $T(c)$ *is consistent with the axioms of set theory.*

Example 3. ($\neg T(\lambda)$) *A Moore space* Λ *of weight* λ *which cannot be embedded into any separable first countable Hausdorff space.*

Corollary 1. The answer YES to the Problem 1 is consistent with and independent of the axioms of set theory.

Corollary 2. The condition $T(\omega_1)$ *is true.*

Corollary 3. The condition $T(c)$ *is consistent with and independent of the axioms of set theory.*

Open Questions

A. Can each normal (resp. metacompact) Moore space of weight $\leq c$ be embedded into a separable Moore space?

B. Can each first countable compact space be embedded into a separable first countable (compact?) space?

The proofs of the above announced results will appear in [DP] and [DP_1]. Additional information concerning embeddings into separable first countable spaces can be found in [P] and the set-theoretic condition $T(\lambda)$ is more thoroughly investigated in [P_1].

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