Research Announcement:

ARCWISE CONNECTED CONTINUA AND THE FIXED POINT PROPERTY

by

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THE FIXED POINT PROPERTY

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A continuum (compact connected metric space) M is *arcwise connected* provided that each pair of distinct points of M is the set of endpoints of at least one arc in M. A topological space X has the *fixed point property* provided that if f:X → X is continuous, then there is a point x ∈ X such that f(x) = x. We are interested in answers to the following:

Question 1. *What are sufficient conditions that an arcwise connected continuum have the fixed point property?*

One condition which is necessary for a 1-dimensional arcwise connected continuum M to have the fixed point property is that M be *uniquely arcwise connected* (i.e. each pair of distinct points is the set of end points of exactly one arc in M). For if M is not uniquely arcwise connected, then M contains a simple closed curve S. Since M is 1-dimensional, M can be retracted onto S [6, p. 83]; S can then be rotated to produce a fixed point free map. In Propositions 1 and 2, although M is only assumed to be arcwise connected, it follows easily from other parts of the hypothesis that M is in fact uniquely arcwise connected. The first of these results is due to Borsuk [3].

Proposition 1. *A continuum which is hereditarily unicoherent and arcwise connected has the fixed point property.*
(If n is a positive integer, then the continuum M is *n-coherent* provided that if A and B are proper subcontinua of M whose union is M, then A ∩ B has at most n components. A 1-coherent continuum is called *unicoherent*. If each subcontinuum of M is n-coherent, then M is *hereditarily n-coherent*.)

An extension of Proposition 1 is the following result, due to Young [8].

Proposition 2.   If M *is an arcwise connected continuum such that each monotone nest of arcs is contained in an arc, then* M *has the fixed point property.*

Corollary.   There exist uniquely arcwise connected continua, with the fixed point property, of each positive dimension.

Proof.   Bing has established [1] that for each positive integer n, there is an hereditarily indecomposable continuum M_n of dimension n.   The cone over M_n satisfies the hypothesis of Proposition 2.

In order for an arcwise connected continuum to have the fixed point property, it is not necessary that it satisfy the condition of Proposition 2, as is shown by the following example.

Example 1.   The Warsaw Circle, W, has the fixed point property.   W is the union of a "sin 1/x curve" and an arc [b,c].
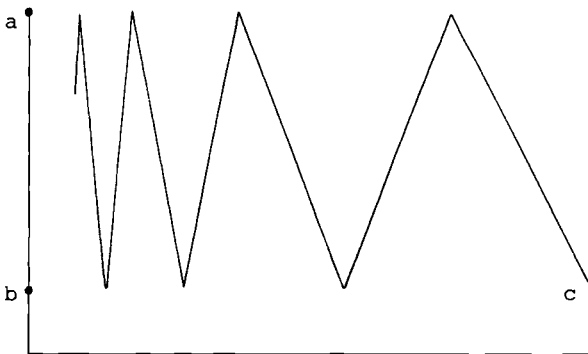

Suppose that f:W → W is a map.   It is easy to see that $\{x : x \in [a, f(x)]\}$ is both open and closed in f[W].   If f[W] is a proper subontinuum of W, it is chainable (i.e. there are open

covers of arbitrarily small mesh whose nerves are arcs) and
so f has a fixed point [4].   Otherwise W = f[W] = {x:x ∈ [a,f(x)]}.
This means that f[a,b] = [a,b], and so f has a fixed point in
[a,b].

The next example, which can be found in [2], [5], and [8]
shows that, for an arcwise connected continuum to have the fixed
point property, something more is needed than the absence of
simple closed curves.

Example 2.   A uniquely arcwise connected continuum M which
does not have the fixed point property.
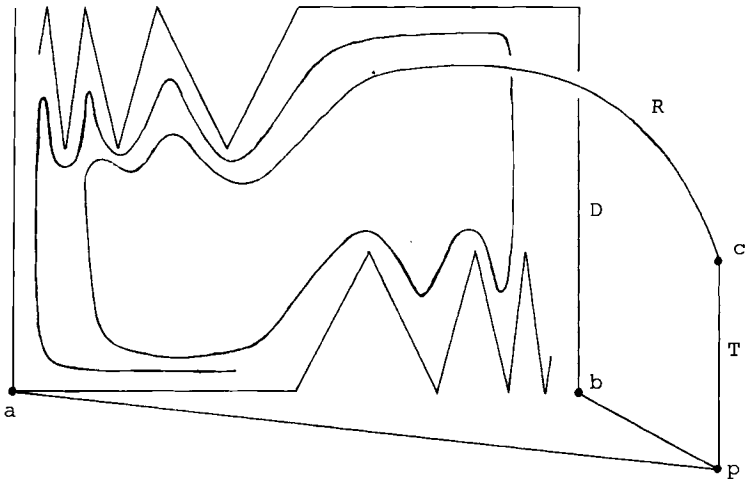
M consists of a "double sin 1/x curve" D, a simple triod T with
the end points a, b and c, and a ray (continuous 1-1 image of
[0,1)) R emanating from c which limits on D, such that
R ∩ (D ∪ T) = {c}.   The map f is a 180° rotation on D, [a,p] is
mapped linearly over [b,c], [b,p] is mapped linearly over [a,c],
and f "stretches" R to "follow the rotation of D."   A detailed
argument can be found in [5].

We now state our principal result.

Theorem 1.   If M *is a uniquely arcwise connected continuum which is hereditarily 2-coherent, and each unicoherent subcontinuum of* M *is hereditarily unicoherent, then* M *has the fixed point property.*

The Warsaw Circle satisfies the hypothesis of this theorem. In fact, it is also a circle-like continuum (i.e. there are finite open covers of arbitrarily small mesh whose nerves are circles). As Nadler [7] points out, there are uncountably many topologically distinct continua which are arcwise connected and circle-like. A corollary of Theorem 1 says that all but one of these has the fixed point property.

Corollary.   If M *is an arcwise connected circle-like continuum, then* M *has the fixed point property if and only if* M *is not a simple closed curve.*

Proof.   Suppose that M is not a simple closed curve. Each proper subcontinuum of a circle-like continuum is chainable, and thus hereditarily unicoherent. It follows that M contains no simple closed curve, and hence is uniquely arcwise connected. Also, each unicoherent subcontinuum of M is hereditarily unicoherent.   From the fact that M is circle-like, we can show that M is 2-coherent and so M is hereditarily 2-coherent.   Now we apply Theorem 1.

There is a point in the proof of Theorem 1 where one needs the following fact: if X is an hereditarily n-coherent continuum and R is a ray which is dense in X, then Int R $\neq \emptyset$. This suggests the following question.

Question 2.   Is there an hereditarily decomposable and uniquely arcwise connected continuum which contains a dense ray with void interior?

Finally, one may ask

Question 3. *Is there an arcwise connected continuum* M,
containing exactly one simple closed curve, which has the fixed
point property?

By the remark at the beginning of this article, dim $M > 1$.
Moreover, no such continuum can be imbedded in a plane, since a
2-dimensional subset of the plane must contain a disk.

References

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