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Research Announcement:

ON FREE TOPOLOGICAL GROUPS AND FREE PRODUCTS OF TOPOLOGICAL GROUPS

by

Temple H. Fay and Barbara Smith Thomas

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Web: http://topology.auburn.edu/tp/

Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA

E-mail: topolog@auburn.edu

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ON FREE TOPOLOGICAL GROUPS AND FREE PRODUCTS OF TOPOLOGICAL GROUPS

(summary of results to appear elsewhere)

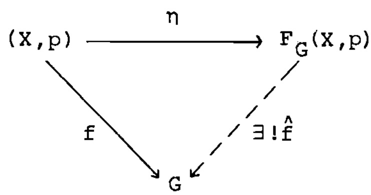
Temple H. Fay and Barbara Smith Thomas

In attempting to characterize the epimorphisms in the category of Hausdorff topological groups, one is led to investigating certain quotients of the free product $G \amalg G$ of a Hausdorff topological group with itself. In particular, one wants to know if the amalgamated free product $G \amalg_B G$, for a closed subgroup B, is Hausdorff, or if its Hausdorff reflection is "sufficiently large." To do this it seems useful to obtain information about the topological structure of $G \amalg G$, or more generally of $G \amalg H$ for Hausdorff topological groups G and H.

Given any two topological groups (not necessarily Hausdorff) G and H, their coproduct $G \amalg H$ in the category of topological groups is $G * H$ (their free product in the category of groups) with the finest topology compatible with the group structure making the injections $G \to G \amalg H \leftarrow H$ continuous (Wyler). The subgroup K of $G \amalg H$ generated by the reciprocal commutators $[g,h] = ghg^{-1}h^{-1}$ is a normal subgroup and is freely generated. It is not difficult to show that every element of $G \amalg H$ has a unique representation ghc with $c \in K$, and if G and H are Hausdorff then K is closed in $G \amalg H$. Theorem 4 below shows that if G and H are Hausdorff and if K can be given a suitable topology, then $G \amalg H$ is Hausdorff.

Since K is freely generated we begin with a short investigation of Graev free groups.

Defn: The *Graev free topological group* over a pointed space (X,p) consists of a topological group $F_G(X,p)$ and a base point preserving continuous function $\eta:(X,p) \to F_G(X,p)$ with the usual unique factorization property



(the base point of a group is the identity, f is continuous and preserves base points, the induced $\hat{f}$ is a continuous group homomorphism, and the diagram commutes).

Theorem (Wyler)*: The underlying group of* $F_G(X,p)$ *is the free group on* $X\backslash\{p\}$*, and* $F_G(X,p)$ *has the finest topology compatible with the group structure such that* $\eta: X \to F_G(X,p)$ *is continuous.*

Theorem (Ordman)*: If* X *is a* k_ω*-space*[1] *then* $F_G(X,p)$ *has the weak topology generated by the subsets* $[F_G(X,p)]_n$ = *words of reduced length* $\leq$ n.

Theorem 1*:* $F_G(X,p)$ *is Hausdorff if and only if* X *is functionally Hausdorff.*

Theorem 2*:* $F_G(X,p)$ *contains* X *as a closed subspace if and only if* X *is Tychonoff.*

Theorem 3*: If* X *is Tychonoff and* Y *is a closed subspace of* X *containing the base point then the subgroup of* $F_G(X,p)$ *generated by* Y *is closed.*

We now turn to considering the topological structure of $G \amalg H$ for Hausdorff groups G and H.

Theorem 4*: In order that* $G \amalg H$ *have a Hausdorff group*

[1] X is a k_ω-space if it is the weak sum of countably many compact Hausdorff spaces.

topology it is necessary and sufficient that $K \lhd G \amalg H$ *have a Hausdorff group topology, no finer than, but comparable to, the topology of* $F_G(G \wedge H, \bar{e})$[2] *such that* $\psi: (G \times H) \times K \to K$, *where* $\psi(g,h,c) = ghch^{-1}g^{-1}$, *is continuous.*

Note: the need for the continuity of ψ shows up a number of times in the proof of this theorem, of which the simplest example is $(ghc)^{-1} = g^{-1}h^{-1}(hgh^{-1}g^{-1})(ghc^{-1}h^{-1}g^{-1})$

$$= g^{-1}h^{-1}[g,h]^{-1}\psi(g,h,c^{-1})$$

Ordman's Theorem above says roughly "If X is a k_ω-space and something is true for words of length $\leq n$, for all n, then it is true for $F_G(X,p)$. Thus

Corollary (Katz)*: If* G *and* H *are* k_ω*-spaces, then* $G \amalg H$ *is Hausdorff.*

The essential observation in the proof is that for all n $\psi_n: (G \times H) \times [F_G(G \wedge H, \bar{e}]_n \to F_G(G \wedge H, \bar{e})$ is continuous.

We conclude with a few observations. The first answers a question of Morris, Ordman, and Thompson in the negative.

Observation 1: $G \wedge H$ and $[G,H] \subseteq G \amalg H$ need not have the same topology: In $X = (\omega_1+1) \times \omega_1$ the two closed sets $A = \{(x,x) | \omega_0 \leq x < \omega_1\}$ and $B = ((\omega_1+1) \times \{1\}) \cup (\{\omega_1\} \times \omega_1)$ cannot be separated by open sets. Let $G = F_G(\omega_1+1, \omega_1)$ and $H = F_G(\omega_1, 1)$. Then X is a closed subset of $G \times H$, A is closed in $G \times H$ and $B = X \cap ((G \times \{e_H\}) \cup (\{e_G\} \times H))$. Hence A and $(G \times \{e_H\}) \cup (\{e_G\} \times H)$ cannot be separated by open sets. It follows that $G \wedge H$ is not even regular, and thus cannot be a subspace of $G \amalg H$.

Observation 2: The method of proof used in the corollary

[2] $G \wedge H$ is the quotient of $G \times H$ obtained by collapsing $(G \times \{e_H\}) \cup (\{e_G\} \times H)$ to a point, this collapsed point is denoted by $\bar{e}$.

above cannot be used in the general case since $F_G(\mathbf{Q},0)$ does not have the weak topology generated by the words of length $\leq n$, and $\mathbf{Q} = \mathbf{Q} \wedge \{0,1\}$.

Obersvation 3: We cannot even replace $F_G(G \wedge H,\bar{e})$ by $F_G(\beta(G \wedge H),\bar{e})$ to obtain a proof of the general case since $(\mathbf{Q} \times \mathbf{Q}) \times F_G(\beta(\mathbf{Q} \wedge \mathbf{Q}),\bar{0})$ does not have the weak topology generated by the subsets $(\mathbf{Q} \times \mathbf{Q}) \times [F_G(\beta(\mathbf{Q} \wedge \mathbf{Q}),\bar{0}]_n$.

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University of Cape Town

Cape Town, South Africa

Memphis State University

Memphis, Tennessee 38112