TOPOLOGY PROCEEDINGS

Volume 1, 1976

Pages 355–365

http://topology.auburn.edu/tp/

PROBLEM SECTION

Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: 0146-4124

PROBLEM SECTION

In each volume of TOPOLOGY PROCEEDINGS, there will be a section containing unsolved problems posed by topologists at the conference. The main source of problems is the papers themselves, but the authors of papers are also asked to submit unsolved problems related to the material in their papers.

In addition, problems from other sources will appear in the problem section if they are of sufficient interest to general topologists. Among other things, we hope eventually to sift through the problems in Mary Ellen Rudin's booklet, *Lectures in Set Theoretic Topology*, [Regional Conference Series in Mathematics No. 23, published by the AMS in 1975] giving references to the solutions and partial solutions, and repeating those unsolved ones which are nontrivial and interesting enough. In the sub-section "Classic Problems," we will restate problems A11, B3, H11, H16, and H18 of the booklet: the first two in Classic Problem I, the third in Classic Problem II, and the last two in III. We invite our readers to send problems that should be included in this sub-section.

Future volumes will contain solutions (complete or partial) to problems posed here. They will also contain references giving a fuller background to problems appearing in earlier volumes. Solutions and references should be submitted to Peter J. Nyikos, Auburn University, Department of Mathematics, Auburn, Alabama 36830. It does not matter whether the person submitting them is the one they are due to, so long as the original source is properly credited.

Before each problem there appears in parentheses the name of the author whose article in this volume contains the question or contains material closely related to it.

A. CARDINAL INVARIANTS

1. (*Kunen*) Does MA + $\neg$CH imply that there are no L-spaces?

2. (*Bankston*) Let ϕ be a cardinal invariant. ϕ is *elementary* if for any space X and ultrafilter D, if $\phi(X) < \aleph_0$ then $\phi(\Pi_D(X)) = \phi(X)$. [Examples are cardinality and weight. Non-elementary invariants are also known.] Find interesting conditions sufficient for a cardinal invariant to be elementary.

See also Problems B1, C1, C2, E2, and Classic Problem II.

B. GENERALIZED METRIC SPACES AND METRIZATION

1. (*Przymusinski*) Can each normal (or metacompact) Moore space of weight $\leq$ c be embedded into a separable Moore space?

2. (*Burke*) Is the perfect image of a quasi-developable space also quasi-developable?

3. (*Alster and Zenor*) Is every locally connected and locally peripherally compact normal Moore space metrizable?

See also Classic Problems II and IV.

C. COMPACTNESS AND GENERALIZATIONS

1. (*Przymusinski*) Can each first countable compact space be embedded into a separable first countable space? A separable first countable compact space?

2. (*Woods*) Is it consistent that there exists a normal countably compact Hausdorff F-space X such that $|C^*(X)| = 2^{\aleph_0}$ and X is not compact?

See also Classic Problem I.

D. PARACOMPACTNESS AND GENERALIZATIONS

1. (*Woods*) Is there a "real" (i.e. not using any set-theoretic hypotheses other than ZFC) example of an extremally disconnected locally compact normal non-paracompact Hausdorff space?

2. (*Smith*) Let X be a regular q-space. If X is $\aleph$-ppc and weakly θ-refinable, then is X paracompact?

3. (*Smith*) Are $\aleph$-ppc or ppc spaces countably paracompact or expandable?

4. (*Smith*) What class of spaces, weaker than irreducible spaces, imply paracompactness in the presence of $\aleph$-ppc?

5. (*Bankston*) Can an ultrapower of a (paracompact) space be normal without being paracompact?

6. (*Alster and Zenor*) Is every perfectly normal, locally Euclidean space collectionwise normal?

7. (*Alster and Zenor*) Is every locally compact and locally connected normal T_2-space collectionwise normal with respect to compact sets?

See also E1 and Classic Problem III.

E. SEPARATION AND DISCONNECTEDNESS

1. (*Wage*) Is there an extremally disconnected Dowker space?

2. (*Wage*) Is there a strong S-space that is extremally disconnected?

3. (*Bankston, attributed to R. Button*) Are ultraproducts of scattered Hausdorff spaces scattered? [Non-Hausdorff counter-examples are known.]

See also D1.

F. CONTINUA THEORY

1. (*Hagopian, attributed to Bing*) Is there a homogeneous tree-like continuum that contains an arc?

2. (*Ingram*) Is there an atriodic tree-like continuum which cannot be embedded in the plane?

3. (*Ingram*) What characterizes the tree-like continua which can be embedded in the plane?

4. (*Ingram*) What characterizes the tree-like continua which are in Class W?

5. (*Gordh and Lum*) Let M be a continuum containing a fixed point p. Are the following conditions equivalent?

(a) Each subcontinuum of M which is irreducible from p to some other point is a monotone retract of M.

(b) Each subcontinuum of M which contains p is a monotone retract of M.

See also the following section.

G. MAPPINGS OF CONTINUA AND EUCLIDEAN SPACES

1. (*Mauldin and Brechner*) Let K be a locally connected, non-separating continuum in E^2, K not a disk. Let h be an EC homeomorphism of K onto itself such that h is extendable to a homeomorphism $\tilde{h}$ of E^2 onto itself. Is h necessarily periodic? Does there exist a homeomorphism $g:E^2 \to E^2$ such that $g\tilde{h}:E^2 \to E^2$ is EC^+ with nucleus K?

2. (*Mauldin and Brechner*) Let h be an orientation preserving, EC^+ homeomorphism of E^2 onto itself. If the nucleus of h is unbounded, can h be imbedded in a flow?

3. (*Mauldin and Brechner*) Characterize the EC and EC^+ homeomorphisms of R^∞.

4. (*Mauldin and Brechner*) Characterize the nuclei of the EC^+ homeomorphisms of E^n and characterize the action of such homeomorphisms on its nucleus.

5. (*Mauldin and Brechner*) Let h be an orientation preserving EC^+ homeomorphism of E^n onto itself whose nucleus M is bounded. If n is 4 or 5, is it true that $\tilde{h}:E^n/M \twoheadrightarrow E^n/M$ is a topological standard contraction?

6. (*Petrus*) Let X be a continuum and let $\mu:C(X) \to [0,\infty)$ be a Whitney map. If $\mu^{-1}(t_0)$ is decomposable, must $\mu^{-1}(t)$ be decomposable for all $t \in [t_0,\mu(X)]$?

7. (*Petrus*) Let $\mu: C(X) \to [0,\infty)$ be a Whitney map. Characterize those continua which satisfy, for all $t \in [0,\mu(X)]$

(a) If $\mathfrak{A}$ is a subcontinuum of $\mu^{-1}(t)$ and

$\sigma\mathfrak{A}$ (that is, $\cup\{A : A \in \mathfrak{A}\}$) $= X$, then $\mathfrak{A} = \mu^{-1}(t)$.

and those which satisfy

(b) If $\mathfrak{A}$ is a subcontinuum of $\mu^{-1}(t)$, then $A \in \mathfrak{A}$ for all $A \in \mu^{-1}(t)$ such that $A \subset \sigma\mathfrak{A}$

H. MAPPINGS OF OTHER SPACES

1. (*Nyikos*) Is there any "reasonably large" class of spaces X, Y for which

$$\text{ind } X \leq \text{ind } Y + n$$

when $f: X \to Y$ is a perfect mapping and

$$\text{ind } f^{-1}(y) \leq n \text{ for all } y \in Y?$$

Does it even hold for all metric spaces? Does it hold if ind Y = 0?

See also B2.

I. INFINITE-DIMENSIONAL TOPOLOGY

1. (*West*) Let G be a compact, connected Lie group acting on itself by left translation. Is $2^G/G$ a Hilbert cube?

2. (*West*) Give conditions ensuring that, if G is a compact Lie group acting on a Peano continuum X, the induced G action on the Hilbert cube 2^X is conjugate to some standard, such as the induced translative action on 2^G.

3. (*West*) In general, given a compact Lie group, give conditions on G actions on manifolds, ANR's, Peano continua, or any other class of spaces which ensure that the induced G actions on hyperspaces are conjugate.

4. (*West*) Let G be a compact Lie group acting on a Peano continuum X and consider the injection of $X \to 2^X$ as the singletons. Then G acts on 2^X and we can iterate the procedure,

obtaining a direct sequence $X \to 2^X \to 2^{(2^X)} \to \cdots$. If we give X a G-invariant, convex metric then the inclusions are isometries, and, moreover, the Hausdorff metric is both G-invariant and convex. Using the expansion homotopies $A \mapsto N_t(A)$, we see that X is a Z-set in 2^X. If we now take the direct limit, we obtain a space which is homeomorphic to separable Hilbert space equipped with the bounded-weak topology and has an induced G action on it. Identify this action directly in terms of ℓ_2.

5. (*West*) If, in the situation of Problem 4, we take the *metric* direct limit, we have a separable metric space with a G action on it.

(a) Characterize this space and/or its completion in terms of more familiar objects. In particular, are they homeomorphic to any well-known vector spaces?

(b) Once (a) is done, characterize the induced G action.

CLASSIC PROBLEMS

A portion of each Problem Section will be given over to problems which for one reason or another have become "classics" of modern general topology. Such, for example, are the normal Moore space conjecture, the S versus L space problem, and the question of whether there are P-points in $\beta N-N$. Since Mary Ellen Rudin's *Lecture Notes in Set Theoretic Topology* remains an excellent and reasonably up-to-date reference to these problems, a detailed treatment of them has been deferred to future issues. Two other acknowledged "classics" have been omitted for a different reason: they have been solved in the last few months! I refer to the problem of whether $\dim(X \times Y) \leq \dim X + \dim Y$ for completely regular spaces (solved in the negative by Wage and Przymusinski) and the problem of whether every Hausdorff compactification is a Wallman compactification, a negative solution to which has been announced by Ul'janov.

I. Does every compact space contain either a nontrivial convergent sequence or a copy of βN? [In this problem only, "compact" will mean "infinite compact Hausdorff."]

Equivalent problems. Does every compact space contain a copy of $\omega + 1$ or a copy of $\beta N - N$? a closed metric subspace or an infinite discrete C*-embedded subspace?

Related problems. Does every totally disconnected compact space contain either a copy of $\omega + 1$ or a copy of βN? Equivalently: Does an infinite Boolean algebra have either a countable infinite or a complete infinite homomorphic image?

Does every compact space contain either a point with a countable π-base or a copy of $\beta N - N$?

Does every compact hereditarily normal space contain a nontrivial convergent sequence? a point with a countable π-base? a point with a countable Δ-base?

(A π-base at a point x is a collection of open sets such that every neighborhood of x contains one; a Δ-base at x is a π-base at x such that every member has x in its closure.)

Consistency results. Assuming CH, Fedorčuk constructed a compact space of cardinal 2^c so that every infinite closed subspace is of positive dimension. Since both $\omega + 1$ and βN are zero-dimensional, this space cannot contain a copy of either one. Assuming V = L, Fedorčuk constructed a space having all the above properites of his first space which is, in addition, hereditarily separable and hereditarily normal.

References.

[1] B. A. Efimov, *On embedding of Stone-Čech compactifications of discrete spaces in bicompacta,* Dok. Ak. Nauk SSSR 189 (1969), 244-246 = Sov. Math. Dok. 10 (1969), 1391-1394.

[2] V. V. Fedorčuk, *Bicompacta in which each infinite closed subset is n-dimensional*, Math. USSR Sbornik 25 (1975), 37-57.

[3] V. V. Fedorčuk, *On the cardinality of hereditarily separable compact Hausdorff spaces*, Dok. Ak. Nauk SSSR 222 (1975), 302-305 = Sov. Math. Dok. 16 (1975), 651-655.

[4] B. E. Šapirovskii, *On π-character and π-weight in compact Hausdorff spaces*, Dok. Ak. Nauk SSSR 223 (1975), 799-802 = Sov. Math. Dok. 16 (1975), 999-1000.

[5] B. E. Šapirovskii, *On the cardinality of hereditarily normal spaces*, Dok. Ak. Nauk SSSR 225 (1975), 767-770 = Sov. Math. Dok. 16 (1975), 1541-1546.

II. Is there a nonmetrizable perfectly normal, paracompact space with a point countable base?

Related problems. Which of the following implications holds for perfectly normal spaces with point countable bases:

normal implies collectionwise normal?

collectionwise normal implies paracompact?

paracompact implies metrizable?

non-Archimedian implies metrizable?

Lindelöf implies metrizable?

This last is equivalent to the question of whether every hereditarily Lindelöf regular space with a point countable base is metrizable, and also to whether it is separable. Moreover, [Tall] it is equivalent to the question of whether every first countable regular space which is of countable spread (in other words, every discrete subspace is countable) is separable. Hence it is also equivalent to the question of whether every first countable, hereditarily Lindelöf regular space is hereditarily separable.

Consistency results. A Souslin line, whose existence is independent of the usual axioms of set theory, is a hereditarily Lindelöf (hence perfectly normal) linearly ordered (hence monotonically normal) space which is not metrizable. *H. R. Bennett*: If there exists a Souslin line, there exists one with a point-countable base. *A. V. Arhangel'skii and P. J. Nyikos*: There exists a hereditarily Lindelöf non-Archimedean space (and such a space necessarily has a point-countable base) which is not metrizable if, and only if, there exists a Souslin line. *E. van Douwen, E. D. Tall, and W. A. R. Weiss*: CH implies the existence of a hereditarily Lindelöf space with a point-countable base which is not metrizable. *J. Silver*: MA + $\neg$CH implies the existence of a normal Moore (hence perfectly normal) space with a σ-point-finite base which is not metrizable, hence not collectionwise normal.

References.

[1] A. V. Arhangel'skii, *Mappings and spaces*, Russ. Math. Surveys 21 (1966), 155-162.

[2] V. I. Ponomarev, *Metrizability of a finally compact* p-*space with a point-countable base*, Sov. Math. Dok. 8 (1967), 765-768.

[3] R. E. Hodel, *Some results in metrization theory*, 1950-1972 in: *Topology Conference*, Lecture notes in Mathematics no. 375, Springer-Verlag, 1974.

[4] F. D. Tall, *On the existence of non-metrizable hereditarily Lindelöf spaces with point-countable bases*, Duke Math. J. 41 (1974), 299-304.

[5] E. van Douwen, F. D. Tall, and W. A. R. Weiss, *Non-metrizable hereditarily Lindelöf spaces with point-countable bases from* CH, to appear.

III. Is every screenable normal space paracompact? (A space is *screenable* if every open cover has a σ-disjoint refinement.)

Equivalent problems. Is every screenable normal space countably paracompact? θ-refinable? countably θ-refinable? [*Nagami*: A screenable normal, countably paracompact space is paracompact. For normal spaces, the concepts of countably paracompact, countably metacompact, countably subparacompact and countably θ-refinable are all equivalent.]

Related problems. Is a screenable normal space collectionwise normal? [Note: It is collectionwise Hausdorff.] Is a screenable, collectionwise normal space paracompact? Is a normal space with a σ-disjoint base paracompact? Is a screenable normal space of non-measurable cardinality realcompact?

Is every collectionwise normal, weakly θ-refinable space paracompact? Is every normal weakly θ-refinable space of non-measurable cardinality realcompact? countably paracompact?

Consistency results. None pertaining to screenability. *P. de Caux*: ♣ implies the existence of a weakly θ-refinable, collectionwise normal space of cardinal $\aleph_1$, which is not realcompact and hence not even countably paracompact.

References.

[1] K. Nagami, *Paracompactness and strong screenability*, Nagoya Math. J. 88 (1955), 83-88.

[2] P. de Caux, *A collectionwise normal weakly θ-refinable Dowker space which is neither irreducible nor realcompact*, these PROCEEDINGS.

IV. Does every stratifiable space have a σ-closure-preserving open base? (In other words, is every M_3 space M_1?)

Equivalent problems. Does any point in any stratifiable space have a closure-preserving local base of open sets? A σ-closure-preserving local base of open sets? Does any closed set in a stratifiable space have a closure-preserving (or: σ-closure-preserving) neighborhood base of open sets?

Related problems. Is the closed image of an M_1 space M_1?

Is the perfect image of an M_1 space M_1? Is the closed irreducible image of an M_1 space M_1? Is every closed subspace of an M_1 space M_1? Is every subspace of an M_1 space M_1?

Consistency results. None.

Partial results. G. Gruenhage and H. Junnila: Every stratifiable space is M_2. (An M_2-space is one with a σ-closure preserving quasibasis, a quasibasis being a collection of sets which includes a base for the neighborhoods of each point.) *Gruenhage:* Every σ-discrete stratifiable space is M_1. *C. R. Borges and D. J. Lutzer:* The irreducible perfect image of an M_1 space is M_1.

References.

[1] J. G. Ceder, *Some generalizations of metric spaces*, Pac. J. Math. 11 (1961), 105-125.

[2] C. R. Borges, *On stratifiable spaces*, Pac. J. Math. 17 (1966), 1-16.

[3] D. J. Lutzer, *Semi-metrizable and stratifiable spaces*, Gen. Top. Appl. 1 (1971), 43-48.

[4] C. R. Borges and D. J. Lutzer, *Characterizations and mappings of* M_i *spaces*, pages 34-40 in: *Topology Conference*, Lecture notes in Mathematics no. 375, Springer-Verlag, 1974.

[5] G. Gruenhage, *Stratifiable spaces are* M_2, these PROCEEDINGS.

[6] H. Junnila, *Neighbornets*, to appear.