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COMPACTIFICATIONS: RECENT RESULTS FROM SEVERAL COUNTRIES

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COMPACTIFICATIONS: RECENT RESULTS FROM SEVERAL COUNTRIES

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1. Introduction

The Organizing Committee of this Conference some weeks ago gave me the duty of preparing a survey talk on Compactification Theory. I found this assignment both interesting and flattering. Interesting because it seems to suggest that there is such a thing as Compactification Theory, and flattering because of the apparent implication that I am competent to survey the subject in 60 minutes. In fact I am not, and in any event the Committee and I have been scooped by the recent appearance of a highly competent, scholarly, comprehensive work [11] by Richard Chandler of the University of North Carolina at Raleigh. I believe that since this book is on a topic central to point-set topology, virtually all of us here have seen it already or will see it soon. Accordingly I have arranged that the results I will mention to you today are essentially disjoint from the theorems of [11]. Even after dogmatically excluding from consideration the content of a new book devoted explicitly to the subject assigned to me, incidentally, I have found my principal difficulty in preparing today's remarks to be not the problem of finding an amount of interesting material adequate to fill an hour but rather the problem of selecting a manageable set of

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theorems from the very large number of new and exciting results available. It is a common truism these days, obvious to anyone who has even a passing acquaintance with such surveys as [39] and [8], that general topology is enjoying an explosive success in connection with topological questions which lie partly in the provinces traditionally associated with logic (independence results), set theory and infinite combinatorics. Less obvious but I now believe quite true is the statement that, similarly, Compactification Theory is vital and flourishing.

Here is my proposed plan of procedure. I will begin (in §2) by citing without proof several of the more exciting results which have come to my attention recently. §3 is devoted to a pair of applications in topology of a partition relation of P. Erdős and R. Rado. The second of these, due to Ginsburg and Woods, is new; the first, proved by Juhász in 1971, is included here because the argument is very pretty and deserves greater notoriety. In §4 are presented two results, one very simple (my own invention) and the other deeper and much less obvious (due essentially to R. Frankiewicz and brought to its present elegant state by B. Balcar and P. Simon), confirming our general suspicion that spaces of ultrafilters over discrete spaces which instinct tells us are not homeomorphic are indeed, at least most of the time, not homeomorphic. Finally, §5 contains a brief summary of Turzański's recent results on the thick spaces introduced by Arhangel'skiĭ.

The expression "space," as used here, refers always to a completely regular, Hausdorff topological space.

I am indebted to the following five or six mathematicians for suggestions about the scope and content of these remarks, for pointing out errors or ambiguities in an early version, and for bibliographical assistance: E. K. van Douwen, M. Hušek, J. Roitman, P. Simon, A. K. Steiner, and the referee.

2. Some Recent Results

2.1. Undoubtedly the most spectacular recent development in Compactification Theory is the announcement by the Soviet mathematician V. M. Uljanov [47] of a solution to "Frink's Wallman problem" given in [20]. The literature of partial solutions to this problem is too extensive to describe or summarize here, but I note two recent developments which preceded that of Uljanov: (1) C. Bandt [5] has shown that every compact (Hausdorff) space X such that $wX \leq \omega^+$ is a Wallman compactification of each of its dense subsets; hence, assuming the continuum hypothesis ($\omega^+ = 2^\omega$), every (Hausdorff) compactification of a separable space is a Wallman compactification; and (2) the full and general problem as originally posed has been reduced to the problem for discrete spaces by L. B. Šapiro [40], independently by C. Bandt [5], and independently by A. K. Steiner and E. F. Steiner [42] (that is, if there is a space with a compactification not of Wallman type then there is such a discrete space). The example of Uljanov [47], given in ZFC with no special set-theoretic assumptions, is in consonance with these results. It reads: There is a non-Wallman compactification of the discrete space of cardinality ω^+.

2.2. A. V. Arhangel'skiĭ [3] has raised the question whether there is a countable space X such that $|BX| = 2^{2^\omega}$ for every compactification BX of X. In an interesting paper quite worthy of study, which extends far beyond the natural boundaries of this question, B. A. Efimov [17] furnishes the answer "Yes". For a suitable example one may take, as many of us had suspected for some time, any countable dense subset D of the space $\{0,1\}^{2^\omega}$. Specifically Efimov shows ([17], Corollary 3.3) that if Y is a compact space whose Boolean algebra of regular-open sets is isomorphic to the corresponding algebra of a dyadic space, and if $cf(\pi wY) > \omega$, then there is compact $K \subset Y$ such that $\chi(y,K) \geq \pi wY$ for all $y \in K$; it then follows from a well-known result often called the Čech-Pospíšil theorem that

$$|Y| \geq |K| \geq 2^{\pi wY} .$$

In the example above, of course, we have

$$\pi w(BD) = \pi wD = 2^\omega \quad \text{and} \quad cf(2^\omega) > \omega$$

for all compactifications BD of D.

More recently and by quite different methods, E. K. van Douwen and T. C. Przymusiński [15] have solved the problem posed by Arhangel'skiĭ. Indeed they have defined a countable space X, all but one point of which is isolated, such that $BX \supset \beta(\omega)$ for every compactification BX of X.

I have learned from István Juhász that his student G. Szentmiklóssy, working at about the same time as Efimov and independently, also proved that the countable space D mentioned above provides a solution to Arhangel'skiĭ's problem.

2.3. The question was raised in 1969 by de Groot [26] whether every compact (Hausdorff) space is supercompact.

(A space X is said to be *supercompact* if there is a subbase $\mathcal{S}$ for the topology of X such that every cover of X by elements of $\mathcal{S}$ has a subcover of cardinality 2.) The argument designed by O'Connor [35] to respond affirmatively to de Groot's question for compact metric spaces is said to be flawed and incomplete, but Strok and Szymański [44] have shown that every compact metric space is supercompact. In its full generality the question has been settled (in the negative) only recently by Murray Bell [7]; indeed, Bell shows that if βX is supercompact then X is pseudocompact. More recently van Douwen and van Mill [14] have shown that a number of familiar compact spaces, among them $\beta(\omega)\backslash\omega$ and indeed every infinite compact space in which every convergent sequence is eventually constant, are not supercompact.

2.4. The space $\beta(\omega)$ continues to generate substantial research and difficult questions (see in this connection §4 below). We mention a few of the recent, strong results.

Following [23], we say that a space X is an *F-space* if disjoint cozero-sets in X have disjoint closures.

(a) Assuming [CH], A. Louveau [32] has characterized the compact spaces which are homeomorphic with a subspace of $\beta(\omega)$ as those compact spaces which are totally disconnected F-spaces of weight $\leq 2^{\omega}$.

(b) Assuming [CH], G. Woods [49] has shown that for a subspace A of a compact F-space X such that $|C^*(X)| = 2^{\omega}$, the following conditions are equivalent: $|C^*(A)| = 2^{\omega}$; A is C^*-embedded in X; A is weakly Lindelöf (in the sense that for every open cover $\mathcal{U}$ of A there is countable $\mathcal{V} \subset \mathcal{U}$

such that $\cup\mathcal{V}$ is dense in A). Woods then deduces, again assuming [CH], that a space X is homeomorphic to a C*-embedded subspace of $\beta(\omega)$ if and only if X is an F-space such that $|C^*(X)| = 2^{\omega}$ and βX is zero-dimensional.

One may admire the first of these results of Woods, I think, as much for its form as for its specific content: It shows that, assuming [CH], the question of whether a subspace X of $\beta(\omega)$ is C*-embedded depends not on how X sits inside $\beta(\omega)$ but on topological properties intrinsic to X itself. It is not known whether this statement, or Woods' theorem itself, can be established without assuming [CH].

(c) In an informal list of theorems privately circulated, S. Glazer [25] announced that every closed, separable subspace of $\beta(\omega)$ is a retract of $\beta(\omega)$. We look forward eagerly to a manuscript with a proof of this and many other of his interesting announcements.

2.5. The following two results have been achieved recently by V. Kannan [30] and by M. Rajagopalan [36], respectively: (a) For $1 \leq \xi \leq \omega^+$ and for every zero-dimensional, non-compact separable metric space X there is a sequential compactification of X of sequential order ξ; (b) For $1 \leq \xi < \omega^+$ there is a scattered, sequential, compactification $B(\omega)$ of ω of sequential order ξ--indeed there are 2^{ω} such compactifications of ω, no two homeomorphic. Further, Kannan and Rajagopalan have shown (independently) that the sequence space S_2 of Arens [1], [4] has no sequential compactification whatever.

We remark in passing that there are scattered spaces with no scattered compactification. The first such space, defined by Nyikos [33], [34], has a base of open-and-closed sets; a later example, due to Solomon [41], does not.

3. A Partition Relation

For cardinal numbers α, κ and λ, the "arrow relation" $\alpha \to (\kappa)^2_\lambda$ means that if $[\alpha]^2 = \cup_{i<\lambda} P_i$ then there are $K \subset \alpha$ with $|K| = \kappa$ and $\bar{i} < \lambda$ such that $[K]^2 \subset P_{\bar{i}}$. (Here as usual for a set A we write

$$[A]^2 = \{B \subset A : |B| = 2\}.)$$

In this notation the celebrated theorem of Ramsey [33] takes the form $\omega \to (\omega)^2_2$. We shall use the following arrow relation of Erdős and Rado: If $\alpha \geq \omega$, then $(2^\alpha)^+ \to (\alpha^+)^2_\alpha$.

A *cellular family* in a space X is a family of pairwise disjoint, non-empty, open subsets of X. The *Souslin number* of X, denoted S(X), is defined by the relation

$$S(X) = \min\{\alpha\colon \text{if } \mathcal{U} \text{ is cellular in X then } |\mathcal{U}| < \alpha\}.$$

In connection with questions about cellular families and their cardinalities, many authors prefer to deal with the cellularity of X, denoted cX and defined by the relation

$$cX = \sup\{\alpha\colon \text{there is cellular } \mathcal{U} \text{ in X with } |\mathcal{U}| = \alpha\}.$$

To my taste this concept is inferior to the Souslin number because in most discussions of cellularity it is necessary to consider separately the case in which there is, and the case in which there is not, a cellular family $\mathcal{U}$ in X such that $|\mathcal{U}| = cX$. In any event the relation between the cardinal numbers cX and S(X) is easily determined:

$$\begin{aligned} S(X) &= (cX)^+ \text{ if there is cellular } \mathcal{U} \text{ in X with } |\mathcal{U}| = cX \\ &= cX \quad \text{otherwise.} \end{aligned}$$

For X a space and $\alpha \geq \omega$ we denote by $X_{\leq\alpha}$ the space whose underlying set of points is (that of) X and whose topology is defined by taking as a basis all sets of the form $\cap\mathcal{U}$ with $\mathcal{U}$ a family of open subsets of X such that $|\mathcal{U}| \leq \alpha$.

Theorem 3.1. *Let* $\alpha \geq \omega$ *and* X *a compact space. If* $S(X) \leq \alpha^+$, *then* $S(X_{\leq\alpha}) \leq (2^\alpha)^+$.

Proof. For every non-empty open subset U of $X_{\leq\alpha}$ there is a family $\{V_\eta : \eta < \alpha\}$ of compact subsets of X, closed under finite intersection, such that

$$\emptyset \neq V = \cap_{\eta<\alpha} \operatorname{int}_X V_\eta \subset \cap_{\eta<\alpha} V_\eta \subset U.$$

Thus if the conclusion of the theorem fails there is a cellular family $\{U(\xi) : \xi < (2^\alpha)^+\}$ in $X_{\leq\alpha}$ and, for $\xi < (2^\alpha)^+$, a family $\{V_\eta(\xi) : \eta < \alpha\}$ of compact subsets of X, closed under finite intersection, such that

$$\emptyset \neq V(\xi) = \cap_{\eta<\alpha} \operatorname{int}_X V_\eta(\xi) \subset \cap_{\eta<\alpha} V_\eta(\xi) \subset U(\xi).$$

For $\xi < \xi' < (2^\alpha)^+$ there is $\langle\eta,\eta'\rangle \in \alpha\times\alpha$ such that $V_\eta(\xi) \cap V_{\eta'}(\xi') = \emptyset$. We define

$$P_{\eta,\eta'} = \{\{\xi,\xi'\} \in [(2^\alpha)^+]^2 : V_\eta(\xi) \cap V_{\eta'}(\xi') = \emptyset\}$$

and we use the Erdős-Rado arrow relation $(2^\alpha)^+ \to (\alpha^+)^2_\alpha$ to find $A \subset (2^\alpha)^+$ with $|A| = \alpha^+$ and $\langle\bar\eta,\bar\eta'\rangle \in \alpha\times\alpha$ such that $[A]^2 \subset P_{\bar\eta,\bar\eta'}$. Now for $\xi \in A$ we define

$$W(\xi) = V_{\bar\eta}(\xi) \cap V_{\bar\eta'}(\xi).$$

Then $\{\operatorname{int}_X W(\xi) : \xi \in A\}$ is a cellular family in X of cardinality α^+, contradicting the relation $S(X) \leq \alpha^+$.

The argument just given is from Juhász [29] (Theorem 2.11). It is remarked by Hedrlín [27] in a different but similar context that for every arrow relation $\alpha \to (\kappa)^2_\lambda$ with

$\lambda \geq \omega$ a theorem analogous to 3.1 above is available. Specifically: If X is compact and $S(X) \leq \kappa$, then $S(X_{\leq\lambda}) \leq \alpha$. The arrow relation $(2^{\alpha})^{+} \rightarrow (\alpha^{+})^{2}_{\alpha}$ of Erdős and Rado is proved (for example) in [29] and [13]; for a comprehensive survey of related results, see [19].

Theorem 3.3 below, due to J. Ginsburg and G. Woods, is a more recent application of this same arrow relation. We begin with a simple lemma (which, together with an appropriate converse, is from [12]).

We say that a subset A of X is *bounded* (in X) if $cl_X A$ is compact.

Lemma 3.2. *If* $S(\beta X \backslash X) > \alpha$ *then there is a family* $\{U_\xi : \xi < \alpha\}$ *of unbounded open subsets of* X *such that* $U_\xi \cap U_{\xi'}$ *is bounded in* X *whenever* $\xi < \xi' < \alpha$.

Proof. Let $\{V_\xi : \xi < \alpha\}$ be a faithfully indexed cellular family in $\beta X \backslash X$, let $p_\xi \in V_\xi$ and $\tilde{V}_\xi$ an open subset of βX such that $\tilde{V}_\xi \cap (\beta X \backslash X) = V_\xi$, let continuous $f_\xi : \beta X \rightarrow [0,1]$ satisfy $f_\xi(p_\xi) = 1$ and $f_\xi(p) = 0$ for all $p \in \beta X \backslash \tilde{V}_\xi$, and set $U_\xi = \{x \in X : f_\xi(x) > 1/2\}$. Since $p_\xi \in cl_{\beta X} U_\xi$ (otherwise $f_\xi(p_\xi) \leq 1/2$), the set U_ξ is not bounded in X. If $\xi < \xi' < \alpha$ and $U_\xi \cap U_{\xi'}$ is unbounded then there is $p \in cl_{\beta X}(U_\xi \cap U_{\xi'}) \cap (\beta X \backslash X)$; then $f_\xi(p) \geq 1/2$ and $f_{\xi'}(p) \geq 1/2$ and hence

$$p \in \tilde{V}_\xi \cap \tilde{V}_{\xi'} \cap (\beta X \backslash X) = V_\xi \cap V_{\xi'},$$

a contradiction. Hence the family $\{U_\xi : \xi < \alpha\}$ is as required.

A family $\mathcal{K}$ of compact subsets of X is a *cofinally compact* family if for every compact $A \subset X$ there is $K \in \mathcal{K}$ such that $A \subset K$. We set

$$\kappa X = \min\{|\mathcal{K}| : \mathcal{K} \text{ is a cofinally compact family}\}.$$

Theorem 3.3 (Ginsburg and Woods [24]). *Let* $\alpha \geq \omega$ *and let* X *be a space such that* $\kappa X \leq \alpha$ *and* $S(X) \leq \alpha^+$. *Then* $S(\beta X \setminus X) \leq (2^\alpha)^+$.

Proof. If the result fails then by Lemma 3.2 (with α replaced by $(2^\alpha)^+$) there is a family $\{U_\xi : \xi < (2^\alpha)^+\}$ of unbounded open subsets of X such that $U_\xi \cap U_{\xi'}$ is bounded in X whenever $\xi < \xi' < (2^\alpha)^+$. Let $\{K_\eta : \eta < \alpha\}$ be a cofinally compact family for X and for $\eta < \alpha$ let

$$P_\eta = \{\{\xi, \xi'\} : U_\xi \cap U_{\xi'} \subset K_\eta\}.$$

Since $[(2^\alpha)^+]^2 = \bigcup_{\eta<\alpha} P_\eta$, it follows from the Erdős-Rado arrow relation $(2^\alpha)^+ \rightarrow (\alpha^+)^2_\alpha$ that there are $A \subset (2^\alpha)^+$ with $|A| = \alpha^+$ and $\bar{\eta} < \alpha$ such that $[A]^2 \subset P_{\bar{\eta}}$. Clearly $\{U_\xi \setminus K_{\bar{\eta}} : \xi \in A\}$ is a family of pairwise disjoint, open subsets of X (each non-empty because $cl_X U_\xi \not\subset K_{\bar{\eta}}$); this contradicts the relation $S(X) \leq \alpha^+$.

For locally compact, σ-compact spaces X we have $\kappa X \leq \omega$ and hence, from Theorem 3.3, $S(\beta X \setminus X) \leq (2^\alpha)^+$ whenever $S(X) \leq \alpha$. Ginsburg and Woods [24] show also, by a simple example, that for $\alpha \geq \omega$ there is a space X such that $S(X) = \omega^+$ and $S(\beta X \setminus X) = \alpha$; thus the hypothesis "$\kappa X \leq \alpha$" cannot be omitted in Theorem 3.3.

We remark, paralleling the comment of Hedrlín cited above, that for every arrow relation $\alpha \rightarrow (\kappa)^2_\lambda$ with $\lambda \geq \omega$ a theorem analogous to the result of Ginsburg and Woods is available. Specifically: If $S(X) \leq \kappa$ and $\kappa X \leq \lambda$, then $S(\beta X \setminus X) \leq \alpha$.

4. Spaces X^* with X Discrete

Versions of the Stone-Čech compactification having been introduced as long ago as 1930 (by Tychonoff [46]) and 1937 (by Stone [43] and Čech [10]), it is reasonable to expect that by now the subspaces $X^* = \beta X \setminus X$ of βX would be well-understood. I want to spend a few moments now discussing the following remarkable and embarrassing situation: Even for (different) discrete spaces X it is not known whether or not the spaces X^* may be homeomorphic.

(At the risk of upsetting the present trend of thought I should probably mention here that every space Y is homeomorphic to X^* for appropriate X; this is noted, for example, by Gillman and Jerison [23] (Problem 9K.6) and by Walker [48] (Theorem 4.3). The space X^* does not determine X--that is, non-homeomorphic spaces may have homeomorphic remainders--but X^* does often convey considerable information about X; for several examples, see [28].

As usual, the Stone-Čech compactification of a (discrete) space α is denoted $\beta(\alpha)$, and α^* denotes $\beta(\alpha)\setminus\alpha$. For $A \subset \alpha$ we write

$$A^* = (\mathrm{cl}_{\beta(\alpha)} A) \setminus A$$

and, with the elements of $\beta(\alpha)$ viewed as ultrafilters on α, we write

$$U(\alpha) = \{p \in \beta(\alpha) : |A| = \alpha \text{ for all } A \in p\} ;$$

and for $A \subset \alpha$ we write $\hat{A} = U(\alpha) \cap A^*$.

The following familiar facts about $\beta(\alpha)$ and its topology are proved and amplified in [13] (§7). These will be used below.

4.1. The sets A^* (with $A \subset \alpha$) form a base for α^*; each set A^* is open-and-closed in α^*; if U is open-and-closed in α^* then there is $A \subset \alpha$ such that $U = A^*$; if $A, B \subset \alpha$ then

$$(A \cap B)^* = A^* \cap B^* \quad \text{and} \quad (A \cap B)\hat{} = \hat{A} \cap \hat{B}$$

and hence, in particular,

$$(A \cap B)^* = \emptyset \text{ iff } |A \cap B| < \omega \quad \text{and}$$

$$(A \cap B)\hat{} = \emptyset \text{ iff } |A \cap B| < \alpha.$$

The cardinal numbers dX and $S(X)$ have been determined [12], [13] for various subsets X of α^* for many cardinals $\alpha \geq \omega$, but we note that Baumgartner [6] has defined a model of ZFC in which the cardinal number ω^+ admits no family $\mathcal{A}$ of subsets such that $|\mathcal{A}| = 2^{(\omega^+)}$ and for every two distinct elements A, B of $\mathcal{A}$ we have $|A \cap B| < |A| = \omega^+$. In this model the attractive relation $S(U(\alpha)) = (2^\alpha)^+$, which holds in all models for many cardinals α, fails for $\alpha = \omega^+$; indeed we have $S(U(\omega^+)) \leq 2^{(\omega^+)}$.

It is well-known that $w(\beta(\alpha)) = w(\alpha^*) = w(U(\alpha)) = 2^\alpha$ for all $\alpha \geq \omega$ (see for example [13] (§7)). Thus if $2^\gamma \neq 2^\alpha$ we have $\alpha^* \not\approx \gamma^*$ and $U(\alpha) \not\approx U(\gamma)$. On the other hand the following theorem of Efimov [16] shows that if $2^\gamma = 2^\alpha$ then each of the spaces α^*, γ^*, $U(\alpha)$, $U(\gamma)$ embeds homeomorphically into each of the others. (This of course does not guarantee that any of the spaces in question are homeomorphic.)

Recall that a space X is said to be *extremally disconnected* if $cl_X\, U$ is open for every open subset U of X. It is clear that every discrete space is extremally disconnected, and it is known that X is extremally disconnected if and only if βX is.

Theorem 4.2 (Efimov [16]). *Let* $\alpha \geq \omega$ *and let* X *be a compact, extremally disconnected space such that* $wX \leq 2^{\alpha}$. *Then* X *is (homeomorphic with) a subspace of* $\beta(\alpha)$.

Proof. From Tychonoff's theorem [13] (Theorem 2.41) we have $X \subset [0,1]^{2^{\alpha}}$. There is a continuous function f from $\beta(\alpha)$ onto $[0,1]^{2^{\alpha}}$ and if $K \subset \beta(\alpha)$ is chosen so that $f|K$ is an irreducible function onto X then $f|K$ is a homeomorphism from K onto X (see for example [13] (page 58)).

Corollary 4.3. *If* $2^{\gamma} = 2^{\alpha} \geq \omega$ *then each of the spaces* $\beta(\alpha)$, α^*, $U(\alpha)$, $\beta(\gamma)$, γ^*, $U(\gamma)$ *embeds homeomorphically into each of the others.*

Proof. Since $U(\alpha) \subset \alpha^* \subset \beta(\alpha)$ and $U(\alpha)$ contains a copy of $\beta(\alpha)$ ([13] (Lemma 7.14)) it is enough to show that $\beta(\gamma)$ embeds homeomorphically into $\beta(\alpha)$; this follows from Theorem 4.2.

We note in passing that for $\alpha \geq \omega$ the spaces α^* and $U(\alpha)$ are not extremally disconnected (see [13] (Theorem 4.11)).

We remark also that the system ZFC + MA + $\neg$CH is equiconsistent with ZFC; and it is a theorem in this system that $2^{\kappa} = 2^{\omega}$ whenever $\omega \leq \kappa < 2^{\omega}$. A nice introduction to MA and its topological consequences is given in [39].

If your intuition parallels mine you feel that for different infinite cardinals α and γ (1) the spaces $U(\alpha)$ and $U(\gamma)$ are not homeomorphic, and (2) the spaces α^* and γ^* are not homeomorphic. Concerning (1), we show in 4.5 and 4.6 below (a little more than the fact that) $U(\omega) \not\approx U(\omega^+)$. As for (2), we have from Theorem 4.10 that if α and γ are cardinals such that $\omega \leq \gamma < \alpha$ and $\gamma^* \approx \alpha^*$, then $\gamma = \omega$ and

$\alpha = \omega^+$; whether there is a model of ZFC in which $\omega^* \approx (\omega^+)^*$ is, however, apparently unknown (cf. 4.11 below).

Lemma 4.4. *Let* $\alpha \geq \omega$ *and* $0 < \kappa < \mathrm{cf}(\alpha)$. *Then there is a cellular family* $\mathcal{U}$ *of open-and-closed subsets of* $U(\alpha)$ *such that* $|\mathcal{U}| = \kappa$ *and* $\cup\mathcal{U}$ *is dense in* $U(\alpha)$.

Proof. Let $\{A_\xi : \xi < \kappa\}$ be a pairwise disjoint family of subsets of α such that $\cup_{\xi<\kappa} A_\xi = \alpha$ and $|A_\xi| = \alpha$ for each $\xi < \kappa$, and define $\mathcal{U} = \{\hat{A}_\xi : \xi < \kappa\}$. That $\cup\mathcal{U}$ is dense in $U(\alpha)$ follows from the fact that if $B \subset \alpha$ and $|B| = \alpha$ then since $\kappa < \mathrm{cf}(\alpha)$ there is $\bar{\xi} < \kappa$ such that $|B \cap A_{\bar{\xi}}| = \alpha$, so that

$$\hat{B} \cap (\cup\mathcal{U}) \supset \hat{B} \cap \hat{A}_{\bar{\xi}} = (B \cap A_{\bar{\xi}})^\wedge \neq \emptyset .$$

Theorem 4.5. *Let* $\alpha \geq \omega$ *and* $\gamma \geq \omega$ *and let* $U(\alpha) \approx U(\gamma)$. *Then* $\mathrm{cf}(\alpha) = \mathrm{cf}(\gamma)$.

Proof. Suppose for example that $\mathrm{cf}(\gamma) < \mathrm{cf}(\alpha)$. To show $U(\alpha) \not\approx U(\gamma)$ it is by Lemma 4.4 enough to show that if $\{\hat{A}_\xi : \xi < \mathrm{cf}(\gamma)\}$ is a faithfully indexed cellular family of open-and-closed subsets of $U(\gamma)$ then $\cup\{\hat{A}_\xi : \xi < \mathrm{cf}(\gamma)\}$ is not dense in $U(\gamma)$. Replacing each set A_ξ with $0 < \xi < \mathrm{cf}(\gamma)$ by $A_\xi \setminus \cup_{\eta<\xi} A_\eta$ if necessary, we assume without loss of generality that the family $\{A_\xi : \xi < \mathrm{cf}(\gamma)\}$ is pairwise disjoint. Then there is $B \subset \gamma$ such that $|B| = \gamma$ and $|B \cap A_\xi| < \gamma$ for all $\xi < \mathrm{cf}(\gamma)$, so that $\hat{B} \neq \emptyset$ and

$$\hat{B} \cap (\cup\{\hat{A}_\xi : \xi < \mathrm{cf}(\gamma)\}) = \emptyset .$$

Corollary 4.6. *If* $\omega \leq \gamma < \alpha$ *with* α *regular, then* $U(\gamma) \not\approx U(\alpha)$. *In particular,* $U(\omega) \not\approx U(\omega^+)$.

We turn next to the spaces α^*.

[Editorial remark by the author. At the Baton Rouge Conference in March, 1977 I presented a preliminary version of Lemma 4.8, due to R. Frankiewicz [22]. Subsequently Frankiewicz himself, and independently P. Simon, achieved the sharpened result given here; the proof below is Simon's (letter of June, 1977). It was B. Balcar who introduced κ-scales into the present context and who, together with Simon (and using the initiative of Frankiewicz), proved Theorem 4.10.]

Lemma 4.7. *If* $\omega \leq \kappa \leq \gamma \leq \alpha$ *and* $\alpha^* \approx \kappa^*$, *then there is* λ *such that* $\omega \leq \lambda \leq \kappa$ *and* $\gamma^* \approx \lambda^*$.

Proof. Let h be a homeomorphism from α^* onto κ^*. Since $\gamma \subset \alpha$ the set γ^* is open-and-closed in α^*, so from 4.1 there is $A \subset \kappa$ such that $h[\gamma^*] = A^*$. The desired conclusion then holds with $\lambda = |A|$.

Lemma 4.8. *Let* $\kappa \geq \omega$ *with* $\kappa^* \approx \omega^*$, *and suppose there are cardinals* α *and* γ *such that* $\alpha > \gamma \geq \kappa$ *and* $\alpha^* \approx \gamma^*$. *Then* $\kappa^* \approx (\kappa^+)^*$.

Proof. If there is $\delta \leq \kappa$ such that $\{\beta > \kappa : \beta^* \approx \delta^*\} \neq \emptyset$ then from Lemma 4.7 we have $\delta^* \approx \omega^*$ and then, choosing $\beta > \kappa$ so that $\beta^* \approx \delta^* \approx \omega^*$ we have, again from Lemma 4.7, that $(\kappa^+)^* \approx \omega^* \approx \kappa^*$. Thus we have $\gamma > \kappa$ and we may assume without loss of generality that $\gamma = \min\{\delta : \{\beta > \kappa : \beta > \delta$ and $\beta^* \approx \delta^*\} \neq \emptyset\}$. We assume further that $\alpha = \min\{\beta > \gamma : \beta^* \approx \gamma^*\}$ and we note that $\alpha = \gamma^+$; indeed otherwise $\gamma^+ < \alpha$ and from Lemma 4.7 there is $\delta \leq \kappa$ such that $(\gamma^+)^* \approx \delta^*$, a contradiction.

Let h be a homeomorphism from α^* onto γ^*. For $\xi < \delta$

the set ξ^* is open-and-closed in γ^*, so from 4.1 there is $A_\xi \subset \alpha$ such that $h[A_\xi^*] = \xi^*$. Since $\gamma = \min\{\delta : \alpha^* \approx \delta^*\}$ and $|\xi| < \gamma$, we have $|A_\xi| < \alpha$. We set $A = \cup_{\xi<\gamma} A_\xi$ and we note that $|A| \leq \gamma \cdot \gamma = \gamma$ and hence there is $\eta < \alpha$ such that $A \subset \eta$.

Since $(\alpha\backslash\eta)^*$ is open-and-closed in α^* there is $B \subset \gamma$ such that $h[(\alpha\backslash\eta)^*] = B^*$. From

$$(\alpha\backslash\eta)^* \cap A^* = \emptyset$$

we have

$$h[(\alpha\backslash\eta)^*] \cap h[A^*] = \emptyset \text{ and hence}$$

$$h[(\alpha\backslash\eta)^*] \cap h[A_\xi^*] = \emptyset$$

for $\xi < \gamma$, i.e., $B^* \cap \xi^* = \emptyset$; it follows that $|B| = \omega$ and hence

$$\alpha^* \approx (\alpha\backslash\eta)^* \approx \omega^*.$$

It follows from Lemma 4.7 that $(\kappa^+)^* \approx \omega^* \approx \kappa^*$, as required. The proof is complete.

Now let $\leq$ and $<$ denote the relations defined on ω^ω by the rules

$$f \leq g \text{ if } |\{n < \omega : f(n) > g(n)\}| < \omega, \text{ and}$$

$$f < g \text{ if } |\{n < \omega : f(n) \geq g(n)\}| < \omega.$$

A subset $\{f_\xi : \xi < \kappa\}$ of ω^ω is a κ-*scale* if

(i) $f_\xi \leq f_\eta$ whenever $\xi < \eta < \kappa$, and

(ii) for all $g \in \omega^\omega$ there is $\xi < \kappa$ such that $g < f_\xi$.

We note that if $\{f_\xi : \xi < \kappa\}$ is a κ-scale in ω^ω and $\{g_\eta : \eta < \lambda\}$ is a λ-scale in ω^ω with κ and λ regular cardinals, then $\kappa = \lambda$. Indeed suppose that $\kappa < \lambda$, for $\xi < \kappa$ choose $\eta(\xi) < \lambda$ such that $f_\xi < g_{\eta(\xi)}$, and let $\bar{\eta} = \sup\{\eta(\xi) : \xi < \kappa\}$; then $\bar{\eta} < \lambda$, and there is no $\xi < \kappa$ such that $g_{\bar{\eta}} < f_\xi$.

Lemma 4.9. *If* $cf(\kappa) > \omega$ *and* $\kappa^* \approx \omega^*$, *then there is a* κ*-scale in* ω^ω.

Proof. Let h be a homeomorphism from κ^* onto ω^*, let $\{A_n : n < \omega\}$ be a partition of κ such that $|A_n| = \kappa$ for $n < \omega$, and (from 4.1) for $n < \omega$ and $\xi < \kappa$ let B_n and C_ξ be subsets of ω such that

$$B_n^* = h[A_n^*] \quad \text{and} \quad C_\xi^* = h[(\kappa\backslash\xi)^*];$$

we assume without loss of generality that $\{B_n : n < \omega\}$ is a partition of ω.

Since $|A_n \cap (\kappa\backslash\xi)| = \kappa$ for $n < \omega$ and $\xi < \kappa$ we have

$$B_n^* \cap C_\xi^* = h[A_n^*] \cap h[(\kappa\backslash\xi)^*] = h[A_n^* \cap (\kappa\backslash\xi)^*] \neq \emptyset$$

and hence $|B_n \cap C_\xi| = \omega$. We define $f_\xi \in \omega^\omega$ by the rule

$$f_\xi(n) = \min(B_n \cap C_\xi)$$

and we claim that $\{f_\xi : \xi < \kappa\}$ is a κ-scale in ω^ω.

(i) Let $\xi < \eta < \kappa$, define $B = \{n < \omega : f_\xi(n) > f_\eta(n)\}$, and suppose $|B| = \omega$. Since $B_n \cap B_{n'} = \emptyset$ whenever $n < n' < \omega$, the function f_η is one-to-one. Thus since $f_\eta(n) \in C_\eta \backslash C_\xi$ whenever $n \in B$ we have $|C_\eta \backslash C_\xi| = \omega$ and hence $C_\eta^* \not\subset C_\xi^*$; this contradicts the relation $\kappa\backslash\eta \subset \kappa\backslash\xi$. It follows that $|B| < \omega$, so that $f_\xi \leq f_\eta$.

(ii) Let $g \in \omega^\omega$. Define

$$C = \cup_{n<\omega}\{k \in B_n : |B_n \cap k| \leq g(n)\}.$$

Then $C^* \neq \emptyset$ (since C contains, for each $n < \omega$, the least element of B_n), and $C^* \cap B_n^* = \emptyset$ for $n < \omega$ (since $|C \cap B_n| \leq g(n) + 1$). Hence, choosing (by 4.1) a subset D of κ such that $h[D^*] = C^*$, we have $D^* \neq \emptyset$ and $D^* \cap A_n^* = \emptyset$ for $n < \omega$. It follows that $|D| = \omega$, so from the hypothesis $cf(\kappa) > \omega$ there is $\bar{\xi} < \kappa$ such that $D \subset \bar{\xi}$. To complete the proof it is enough to show that $g < f_{\bar{\xi}}$.

Let $E = \{n < \omega : g(n) \geq f_{\overline{\xi}}(n)\}$ and suppose $|E| = \omega$. For $n \in E$ we have $f_{\overline{\xi}}(n) \in B_n \cap C_{\overline{\xi}}$ and also $f_{\overline{\xi}}(n) \in C$ (since $|B_n \cap f_{\overline{\xi}}(n)| \leq f_{\overline{\xi}}(n) \leq g(n))$. Since $f_{\overline{\xi}}$ is one-to-one, from $|E| = \omega$ we have $|C_{\overline{\xi}} \cap C| = \omega$ and hence $C^*_{\overline{\xi}} \cap C^* \neq \emptyset$; this contradicts the relation

$$\begin{aligned} C^*_{\overline{\xi}} \cap C^* &= h[(\kappa\backslash\overline{\xi})^*] \cap h[D^*] = h[(\kappa\backslash\overline{\xi})^* \cap D^*] \\ &= h[((\kappa\backslash\overline{\xi}) \cap D)^*] \\ &= h[\emptyset] = \emptyset . \end{aligned}$$

Theorem 4.10. *Let* $\alpha > \gamma \geq \omega$ *and* $\alpha^* \approx \gamma^*$. *Then* $\gamma = \omega$ *and* $\alpha = \omega^+$.

Proof. We note from Lemma 4.8 (with $\kappa = \omega$) that $\omega^* \approx (\omega^+)^*$.

We show first that $\gamma = \omega$. If not then again from Lemma 4.8 (with $\kappa = \omega^+$) we have $(\omega^+)^* \approx (\omega^{++})^*$. Hence from Lemma 4.9 there are an ω^+-scale and an ω^{++}-scale in ω^ω, and from the remark preceding the statement of Lemma 4.9 we have $\omega^+ = \omega^{++}$. This contradiction shows that $\gamma = \omega$.

Suppose now that $\alpha > \omega^+$. Then from Lemma 4.7 (with $\langle\alpha,\omega^{++},\omega\rangle$ in the role of $\langle\alpha,\gamma,\kappa\rangle$ in the statement of Lemma 4.7) we have $(\omega^{++})^* \approx \omega^*$ and hence, again from Lemma 4.9 and the remarks preceding its statement, the contradiction $\omega^+ = \omega^{++}$. The proof is complete.

4.11. *Some Questions and Commentary*. The only question not answered above concerning homeomorphisms between spaces of the form α^* is this.

(a) Is it a theorem in ZFC that $\omega^* \not\approx (\omega^+)^*$? We note that the relation $\omega^* \not\approx (\omega^+)^*$ is valid if the continuum hypothesis $\omega^+ = 2^\omega$ is assumed (since then, as indicated above,

$w(\omega^*) = 2^\omega < 2^{2^\omega} = 2^{(\omega^+)} = w((\omega^+)^*))$. Mary Ellen Rudin in conversation has established the same conclusion in the system ZFC + MA + $\neg$CH. Indeed it is easy to see (in ZFC) that $\{\xi^* : \xi < \omega^+\}$ is an increasing ω^+-sequence in $(\omega^+)^*$ with dense union (so that $\cap_{\xi<\omega^+}(\omega^+\backslash\xi)^*$ has empty interior in $(\omega^+)^*$); but Booth [9] has shown from MA that if $\kappa < 2^\omega$ and $\{A_\xi : \xi < \kappa\}$ is a family of subsets of ω for which each finite subfamily has infinite intersection then there is infinite $A \subset \omega$ such that $|A\backslash A_\xi| < \omega$ for all $\xi < \kappa$ (and hence $\emptyset \neq A^* \subset \cap_{\xi<\kappa} A_\xi^*$). Frankiewicz [21] has deduced the same conclusion, $\omega^* \not\approx (\omega^+)^*$, from property Q of Rothberger [38].

It might appear at first glance that Corollary 4.5 would settle 4.11(a) positively, but in fact it is not clear that a homeomorphism of $U(\omega^+)$ onto itself can be extended to a homeomorphism of $(\omega^+)^*$ onto itself. The following question now comes to mind.

(b) Does every homeomorphism of $(\omega^+)^*$ onto itself take $U(\omega^+)$ to $U(\omega^+)$?

We give now two proofs that a negative answer to (a) yields a negative answer to (b); the first of these occurred to me and the second was communicated informally at this meeting by Eric van Douwen. Further, van Douwen has shown that (b) and (a) are equivalent; the required additional implication is given here with his kind permission.

We note first that $(\omega^+\backslash\omega)^* \approx (\omega^+)^*$. Thus if $(\omega^+)^* \approx \omega^*$ then there is a homeomorphism f from $(\omega^+\backslash\omega)^*$ onto ω^*. Then the function $h : (\omega^+)^* \to (\omega^+)^*$ defined by

$$h|(\omega^+\backslash\omega)^* = f, \; h|\omega^* = f^{-1}$$

is a homeomorphism of $(\omega^+)^*$ onto $(\omega^+)^*$ such that

$$U(\omega^+) \cap h[U(\omega^+)] \subset (\omega^+ \backslash \omega)^* \cap \omega^* = \emptyset .$$

Alternatively, simply use the fact that if $p \in \omega^*$ then

$$\{h(p) : h \text{ is a homeomorphism of } \omega^* \text{ onto } \omega^*\}$$

is dense in ω^*; while if (b) is answered affirmatively then for every $p \in U(\omega^+)$ and every countable, infinite $A \subset \omega^+$ there is no homeomorphism h of $(\omega^+)^*$ onto $(\omega^+)^*$ such that $h(p) \in A^*$.

For the converse we note that if $p \in U(\omega^+)$ then every neighborhood of p in $(\omega^+)^*$ contains a subset which is open-and-closed in $(\omega^+)^*$ and homeomorphic to $(\omega^+)^*$; and if $p \in (\omega^+)^* \backslash U(\omega^+)$ then there is a neighborhood of p in $(\omega^+)^*$ in which each non-empty subset which is open-and-closed in $(\omega^+)^*$ is homeomorphic to ω^*. It follows that if $\omega^* \not\approx (\omega^+)^*$ and $p \in U(\omega^+)$, then $h(p) \in U(\omega^+)$ for each homeomorphism of $(\omega^+)^*$ onto $(\omega^+)^*$.

It follows from Efimov's theorem (4.2 above) that if $2^\kappa = 2^\omega$ then $U(\omega)$ contains a copy of $\beta(\kappa)$--equivalently, there is in $U(\omega)$ a discrete, C*-embedded subset of cardinality κ. On the other hand if $\omega^+ = 2^\omega$ then every discrete, C*-embedded subset D of $U(\omega)$ satisfies $|D| \leq \omega$. This suggests the following question, which has been raised by Woods [49] (Remark 3.4(3)) and, independently in conversation, by van Douwen.

(c) Is there in ZFC a discrete subset D of $U(\omega)$ such that $|D| = \omega^+$ and D is not C*-embedded? We record now a comment, due to K. Prikry and communicated by van Douwen, which is relevant to this question.

K. Kunen [31] defined a model of ZFC in which $\omega^+ < 2^\omega$ and there is $p \in U(\omega)$ such that $\chi(p, U(\omega)) = \omega^+$. In this model the ultrafilter p is a P-point of $U(\omega)$ and $2^\omega = 2^{(\omega^+)}$. Let $\{U_\xi : \xi < \omega^+\}$ be a local base at p such that

$$U_\xi \subset cl_{U(\omega)} U_\xi \subsetneq int_{U(\omega)} \cap_{\eta<\xi} U_\eta \text{ for } \xi < \omega^+,$$

choose $p_\xi \in (int_{U(\omega)} \cap_{\eta<\xi} U_\eta) \backslash cl_{U(\omega)} U_\xi$, and set $A = \{p_\xi : \xi < \omega^+\}$. Then A is a discrete subset of $U(\omega)$ such that $|A| = \omega^+$ and $p \in cl_{U(\omega)} B$ whenever $B \subset A$ and $|B| = \omega^+$; hence the discrete set A is not C*-embedded in $U(\omega)$. On the other hand from Corollary 4.3 (and the relation $2^\omega = 2^{(\omega^+)}$) there are, in this model, discrete and C*-embedded subsets of $U(\omega)$ of cardinality ω^+. Thus in the model of Kunen [31] some but not all discrete subsets of $U(\omega)$ of cardinality ω^+ are C*-embedded.

(d) We have seen above that if $\alpha > \gamma \geq \omega$ and $\alpha^* \approx \gamma^*$ then $\gamma = \omega$ and $\alpha = \omega^+$; and if $\alpha \geq \omega$, $\gamma \geq \omega$ and $cf(\alpha) \neq cf(\gamma)$, then $U(\alpha) \not\approx U(\gamma)$. Encouraged by these results we ask the following question (here as in [13] we write

$$U_\kappa(\alpha) = \{p \in \beta(\alpha) : |A| \geq \kappa \text{ for all } A \in p\} \text{ for}$$

$\omega \leq \kappa \leq \alpha$).

Let $\omega \leq \kappa \leq \alpha$ and $\omega \leq \lambda \leq \gamma$ and let $U_\lambda(\gamma) \approx U_\kappa(\alpha)$. Does it follow that $\gamma = \alpha$ and $\lambda = \kappa$?

5. Thick Spaces

These spaces were introduced by Arhangel'skiǐ [2] in an effort to achieve an internal or intrinsic characterization of dyadic spaces. As Arhangel'skiǐ himself has noted, they don't quite turn the trick: the one-point compactification of the discrete space ω^+ and the one-point compactification of the space ω^+ in its usual order topology are examples of thick spaces which are not dyadic. Nevertheless the class of thick spaces, which contains the class of dyadic spaces [2], [45], has proved interesting in its own right. Following Turzański [45], we note here that certain results known earlier for dyadic spaces are in fact true for the wider class of thick spaces.

Definition. A space X is *thick* if for every $\alpha \geq \omega$ there is dense $D \subset X$ such that

$(*\alpha)$: if $A \subset D$ and $|A| \leq \alpha$, then $cl_D A$ is compact and $w(cl_D A) \leq \alpha$.

(When $(*\alpha)$ holds for D we shall say that D *realizes* $(*\alpha)$ for X.)

Lemma 5.1. *Let* BX *be a thick compactification of* X. *Then* $w(BX) = wX$.

Proof. Of course $wX \leq w(BX)$. Let $\alpha = wX$ and let D realize $(*\alpha)$ for BX. Let $\{U_\xi : \xi < \alpha\}$ be a base for X, for $\xi < \alpha$ let $\tilde{U}_\xi$ be an open subset of BX such that $\tilde{U}_\xi \cap X = U_\xi$, let $x_\xi \in \tilde{U}_\xi \cap D$ and set $A = \{x_\xi : \xi < \alpha\}$. It is then clear that A is dense in BX. Indeed otherwise there are $x \in X$ and a compact neighborhood V of x in BX such that $V \cap A = \emptyset$, and then choosing $\xi < \alpha$ such that $V \cap X \supset U_\xi$ we have

$$(cl_{BX}\ \tilde{U}_\xi) \cap A = (cl_{BX}\ U_\xi) \cap A \subset (cl_{BX}\ V) \cap A$$
$$= V \cap A = \emptyset,$$

a contradiction. Since $cl_D\ A$ is compact we have $cl_D\ A = BX$ and hence $w(BX) = w(cl_D\ A) \leq \alpha$, as required.

Our proof of Corollary 5.2 may be contrasted with the proofs given in [2] and [45].

Corollary 5.2. *Let* X *be a separable metric space and* BX *a compactification of* X. *Then* BX *is thick if and only if* BX *is metrizable.*

Proof. It is well-known (see for example [13], page 134) that every compact metric space is dyadic (and hence thick). Conversely if BX is thick then $w(BX) = wX \leq \omega$ and hence BX is metrizable.

Many of the arguments which serve effectively in the context of thick spaces were introduced by R. Engelking and A. Pelczyński [18]. Our last two results, taken from [45], are of this sort.

We denote by $\mathbb{R}$ the real line in its usual topology, and by $\mathbb{Q}$ the set of rational numbers.

Corollary 5.3. $\beta\mathbb{R}$ *is not thick.*

Proof. It is enough to note that $w\mathbb{R} = \omega$ while $w(\beta\mathbb{R}) > \omega$. (The latter inequality is well-known. It is shown in [23] (Theorem 14.27 and Exercise 14.N.1) that $\beta\mathbb{R}\setminus\mathbb{R}$ is an infinite, compact space in which every convergent sequence is eventually constant; hence $\beta\mathbb{R}\setminus\mathbb{R}$ is not metrizable. Alternatively, note from [12] that $S(\beta\mathbb{R}\setminus\mathbb{R}) \geq (2^{\omega})^{+}$.)

Theorem 5.4. *If* βX *is thick, then* X *is pseudocompact.*

Proof. If X is not pseudocompact then there is continuous $f : X \to \mathbb{R} \subset \beta\mathbb{R}$ such that $\mathbb{Q} \subset f[X]$. The Stone extension $\overline{f}$ of f satisfies $\overline{f}[\beta X] = \beta\mathbb{R}$. Since βX is thick its continuous image $\beta\mathbb{R}$ is thick, contrary to Corollary 5.3.

We note that Theorem 5.4 cannot be strengthened to assert that if some compactification BX of a space X is thick, then X is pseudocompact. For example, let X = ω and BX its one-point compactification.

It is appropriate to close these remarks on thick spaces by reiterating the challenge (the problem) of Arhangel'skiǐ: Find an internal, intrinsic characterization of dyadic spaces.

I thank you for your attention.

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