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# A SURVEY OF $k_\omega$-SPACES

**Stanley P. Franklin and Barbara V. Smith Thomas**

The $k_\omega$-spaces, a natural generalization of countable CW-complexes, have appeared recently in papers on topological groups and topological semigroups. They seem destined to play an important role in that study. Here we survey the purely topological behavior of such spaces. Being unusually well endowed k-spaces they behave very nicely indeed. In particular, they possess excellent separation properties, a nice external characterization, and a very interesting metrization theorem.

Writing $\mathbf{R} = \cup_{n=1}^{\infty}[-n,n]$ expresses the real line as the union of an increasing sequence of compact Hausdorff subsets, with $\mathbf{R}$ having the weak topology with respect to the $[-n,n]$ (i.e., $F$ is closed in $\mathbf{R}$ iff $F \cap [-n,n]$ is closed in $[-n,n]$ for each $n$). Such a decomposition of a space (i.e., $X = \cup_{n=1}^{\infty}X_n$ with the $X_n$ compact Hausdorff and increasing, and $X$ having their weak topology) is called a $k_\omega$-*decomposition*. A space possessing a $k_\omega$-decomposition is called a $k_\omega$-*space*. Hence $\mathbf{R}$ is a $k_\omega$-space.

Caution: If we enumerate the rationals $\mathbf{Q} = \{q_1,q_2,q_3,\cdots,q_n,\cdots\}$ and write $\mathbf{Q}_n = \{q_1,q_2,\cdots,q_n\}$ then we may express $\mathbf{Q} = \cup_{n=1}^{\infty} \mathbf{Q}_n$ as the union of an increasing sequence of compact Hausdorff subspaces but not produce a $k_\omega$-decomposition. The weak topology arising from this sequence is discrete. We shall see later that $\mathbf{Q}$ with its usual topology admits no $k_\omega$-decomposition.

Worse yet can happen. The space $S_2$ (the prototype of sequential but not Fréchet spaces, see [1]) admits two decompositions into the union of an increasing sequence of compact Hausdorff subsets, one of them a $k_\omega$-decomposition, the other not.

The terminology "$k_\omega$-spaces" seems to be due to E. Michael [8]; Graev discussed such spaces with reference to topological groups in 1948 [5], and Morita introduced them under the name $\mathcal{G}'$ in 1956 [10]. The related notion of hemicompactness was introduced by Arens in 1946 [2]. A space, X, is said to be *hemicompact* if it can be written as the union of countably many compact subspaces, $K_n$, with the property that each compact subset of X is contained in a finite union of the $K_n$'s.

Some applications of $k_\omega$-spaces are:

A. If $X = \cup_{n=1}^{\infty} X_n$ is a $k_\omega$-space then so is the Graev free group over X, with $k_\omega$-decomposition $F_G(X,p) = \cup_{n=1}^{\infty}(F_G(X_{n,p}))_n$ (words of reduced length $\leq$ n and letters from $X_n$) (Ordman [12]).

B. If G and H are topological groups which are $k_\omega$-spaces so also is $G \amalg H$ (Katz [6], Ordman [13]).

These results are interesting in that they describe the topological structure of the free group, respectively coproduct, a problem which is unsolved in general.

C. X is $k_\omega$ if and only if C(X) with its compact open topology is completely metrizable (putting together results of Warner [15] and Mosiman and Wheeler [11]).

Ordman (in [12]) has collected some known facts about $k_\omega$-spaces. We state them here, without proof, for the convenience of the reader.

1) (Graev) *Every* $k_\omega$*-space is a Hausdorff* k*-space.*

2) *If* $X = \cup X_n$ *is a* $k_\omega$*-decomposition and* $f: X \to Y$ *is continuous on each* $X_n$*, then* f *is continuous on* X.

3) (Steenrod) *If* $X = \cup X_n$ *is a* $k_\omega$*-decomposition, then each compact subset of* X *is contained in some* $X_n$*. That is,* X *is hemicompact with respect to its* $k_\omega$*-decomposition.*

4) (Milnor) *If* $X = \cup X_n$ *and* $Y = \cup Y_n$ *are* $k_\omega$*-decompositions, so is* $X \times Y = \cup(X_n \times Y_n)$.

Thus $k_\omega$-spaces are finitely productive.

The crux of 3) is that if $A \subset X$ meets every $X \backslash X_n$ and if we let $x_n \in A \cap (X \backslash X_n)$ then $U_m = X \backslash \{x_n | n \geq m\}$ is an open cover of A with no finite subcover. Number 4) is really a theorem about countable CW-complexes, but the same ideas and techniques work for $k_\omega$-spaces.

There's another known and useful fact about $k_\omega$-spaces due to Ordman [13].

5) *In a* $k_\omega$*-space, a point without a compact neighborhood cannot support a countable neighborhood base.*

$k_\omega$-spaces can, in fact, boast of much better separation properties than 1) indicates. If $X = \cup X_n$ is a $k_\omega$-decomposition, and E and F are disjoint closed subsets of X, then $E \cap X_1$ and $F \cap X_1$ have $X_1$ neighborhoods $U_1$ and $V_1$ whose closures are disjoint. Let $E_1 = E \cup \mathrm{cl}U_1$ and $F_1 = F \cup \mathrm{cl}V_1$. Then $E_1 \cap X_2$ and $F_1 \cap X_2$ have $X_2$ neighborhoods $U_2$ and $V_2$ whose closures are disjoint. Continuing, we produce disjoint neighborhoods $U = \cup U_n$ and $V = \cup V_n$ of E and F. Hence X is normal.

In particular we have,

6) (Morita) *Each* $k_\omega$*-space, being* $\sigma$*-compact and regular, is Lindelöf, paracompact, etc.*

We conclude immediately that a $k_\omega$-space with any of several weaker compactness properties (pseudo-compact, countably compact) is already compact. $k_\omega$-spaces are very nice indeed.

Fact 3) above (each compact set contained in some member of a $k_\omega$-decomposition) is not only true of $k_\omega$-decompositions, but actually characterizes them for k-spaces. Suppose $X = \cup X_n$ where X is a k-space and the $X_n$ form an increasing sequence of compact Hausdorff subspaces of X such that each compact subspace of X is contained in one of the $X_n$'s. Suppose also that F meets each $X_n$ in a closed set. Let $K \subseteq X$ be compact. Then $K \subseteq X_n$ for some n and $F \cap K = (F \cap X_n) \cap K$ is closed, and thus F is closed. Stated formally, we have

7) *An increasing sequence of compact Hausdorff subsets covering a* k*-space* X *is a* $k_\omega$*-decomposition iff each compact subset of* X *is contained in one of them.*

An immediate consequence is that

7') *The* $k_\omega$*-spaces are precisely the hemicompact* k*-spaces.*

In 7) we must assume X to be a k-space. Consider $\mathbf{N} \cup \{p\} \subseteq \beta\mathbf{N}$. Let $K_n = \{p, 1, 2, \cdots, n\}$; every compact subset of $\mathbf{N} \cup \{p\}$ is contained in some $K_n$. Thus $N \cup \{p\}$ is hemicompact but fails to be $k_\omega$.

Here's another useful fact about $k_\omega$-decompositions.

8) *Suppose* $X = \cup X_n$ *is a* $k_\omega$*-decomposition and* $X = \cup X'_m$ *is another increasing cover by compact subsets. If each* $X_n$ *is contained in some* $X'_m$, *then the* $X'_m$ *also form a* $k_\omega$*-decomposition, and conversely.*

To see this, suppose F meets each $X'_m$ in a closed set. Given n, choose m so that $X_n \subseteq X'_m$. $F \cap X_n = (F \cap X'_m) \cap X_n$ is closed, and hence F is closed. The converse is immediate from 3).

It follows from 8) that

9) *Any subsequence of a* $k_\omega$*-decomposition is again a* $k_\omega$*-decomposition.*

Now that we have gained some familiarity with the workings of $k_\omega$-spaces it's only natural to ask "what are they?". The obvious conjecture is easily shot down: $\mathbb{Q}$ is a $\sigma$-compact k-space which isn't $k_\omega$ as we'll show below. "But $\mathbb{Q}$ isn't locally compact," you say. Any discrete space of cardinality $> \aleph_0$ is locally compact but not $k_\omega$. "What if we require both?" Now we're getting somewhere.

10) *A locally compact Hausdorff space is* $k_\omega$ *iff it is* $\sigma$*-compact.*

Suppose X is both $\sigma$-compact and locally compact, and let $X = \cup Y_n$ with each $Y_n$ compact. We'll now define a $k_\omega$-decomposition by recursion. Let $X_1 = Y_1$. Cover each point of $X_1$ with a compact neighborhood. Reduce to a finite cover. Union these together and add $Y_2$ to form $X_2$. Thus $X_1 \subseteq X_2$, $Y_2 \subseteq X_2$, and each point of $X_1$ has a compact neighborhood in $X_2$. Continuing in this way we define the subsequent $X_n$. Now to show they form a $k_\omega$-decomposition: suppose F meets each $X_n$ in a closed set and that $x \notin F$. This x belongs to some $X_n$ and hence to int $X_{n+1}$. Since $F \cap X_{n+1}$ is compact, there is an open U containing x and missing $F \cap X_{n+1}$. $U \cap \text{int } X_{n+1}$ is then a neighborhood of x missing F, and F is therefore closed.

The proof of 10) yields yet another sufficient description of a $k_\omega$-decomposition as *an increasing sequence of compact Hausdorff subsets each contained in the interior of a subsequent one.*

One might hope to extend the equivalence of $\sigma$-compact and $k_\omega$ for locally compact spaces to the larger class of k-spaces locally compact on a dense subset. (In the presence of $\sigma$-compactness, these are precisely the Baire spaces.) Unfortunately $\mathbb{Q}$, with the dyadic rationals made discrete, provides the counter example (apply 14).

On the other hand a $k_\omega$-space need not be a Baire space. The space $S_\omega$ ([1]), being $\sigma$-compact and nowhere locally compact, isn't Baire. However, it is the quotient of a countable disjoint sum of convergent sequences (and their limits) and is thus a $k_\omega$-space by 13).

Note that there are $k_\omega$-spaces (like $S_2$) which aren't locally compact, so we haven't yet found them all. But we're close--just two easy lemmas.

Although continuous images of $k_\omega$-spaces need not be $k_\omega$ (map $\mathbb{Q}$ with the discrete topology onto $\mathbb{Q}$ with its usual topology), quotients must be

11) (Morita) *If* $X = \cup X_n$ *is a* $k_\omega$*-decomposition and* $q\colon X \to Y$

*is a quotient map onto a Hausdorff* $Y$, *then the sets* $q(X_n)$ *form a* $k_\omega$*-decomposition for* $Y$.

The proof is straightforward and will be omitted.

12) (Morita) *The disjoint topological sum (coproduct) of countably many* $k_\omega$*-spaces is again a* $k_\omega$*-space*

Let $X^j = \cup X_n^j$ be a $k_\omega$-decomposition for each $j$. Then $\oplus_j X^j = \cup_{m=1}^{\infty}(\cup_{n+j=m} X_n^j)$ is a $k_\omega$-decomposition of the coproduct.

Now given a $k_\omega$-space $X = \cup X_n$ we take the coproduct $\oplus X_n$ of its pieces. The map $i\colon \oplus X_n \to X$ generated by the inclusions $X_n \to X$ is a quotient map since $X$ has the weak topology of the $X_n$. Thus *an arbitrary* $k_\omega$*-space can be represented as the quotient of a countable coproduct of compact Hausdorff spaces*. By 11) and 12) all such quotients of countable coproducts are $k_\omega$-spaces. Thus we have our desired external characterization. Remembering that the k-spaces are precisely the quotient of locally compact spaces, and keeping 10), 11), 12) in mind, we may write,

13) *The* $k_\omega$*-spaces are precisely the quotients of* $\sigma$*-compact locally compact spaces.*

Or as Morita stated it

13') *The* $k_\omega$*-spaces are precisely the quotients of locally compact Lindelöf spaces.*

Now, given the compact Hausdorff spaces, we can construct all $k_\omega$-spaces. Nice, but that still leaves us a lot to know about how they behave. Let's first note some hereditary, and map preservation properties and then go on to look more closely at products.

14) *A closed subspace* Y *of a* $k_\omega$*-space* $X = \cup X_n$ *has a* $k_\omega$*-decomposition* $Y = \cup(X_n \cap Y)$.

The proof is easy. Clearly being a $k_\omega$-space is not arbitrarily hereditary since $\mathbf{Q} \subseteq \mathbf{R}$.

For inverse preservation by mappings we need to go to k-mappings, i.e., maps with the property that inverse images of compact subsets are compact.

15) *If* $Y = \cup Y_n$ *is a* $k_\omega$*-decomposition,* X *is Hausdorff, and* $p\colon X \to Y$ *is an onto* k*-map, then the sets* $p^{-1}(Y_n)$ *form a* $k_\omega$*-decomposition for* X.

Since Y is a k-space, p is a closed map (and hence perfect). Suppose F meets every $p^{-1}(Y_n)$ in a closed set. Then $p(F) \cap Y_n = p(F \cap p^{-1}(Y_n))$ is closed for each n so that p(F) is closed. Thus $\mathrm{cl}F \subseteq p^{-1}p(F)$. If $x \in \mathrm{cl}F\setminus F$, then x is not in the compact set $p^{-1}(p(x)) \cap F$. Thus we may separate these two by disjoint open neighborhoods U and V, i.e., $x \in U$, $p^{-1}(p(x)) \cap F \subseteq V$. Since $x \in \mathrm{cl}F$, U meets F. Thus $x \in \mathrm{cl}F\setminus V$. If we now start the argument anew with $F\setminus V$ instead of F we find that $p^{-1}(p(x))$ must meet $F\setminus V$ which is absurd. Thus F is closed and we are done.

On to products. We have seen in 4) that finite products work. One would immediately suppose that uncountable products won't. What about countable products? Assume that $X_1 \subseteq X_2 \subseteq \cdots$ is a sequence of compact subsets of $\mathbf{R}^{\mathbf{N}}$. Then each $X_i$ is contained in some compact box $\Pi_j[-n_i^j, n_i^j]$. But then the compact box $\Pi_j[-n_j^j-1, n_j^j+1]$ is contained in no $X_i$ contradicting 3). Thus $\mathbf{R}^{\mathbf{N}}$ is not a $k_\omega$-space. So, not all countable products work. Which ones do? Answer: None (except trivial cases).

16) *The product of countably many non-compact* $k_\omega$*-spaces is never* $\sigma$*-compact.*

Write the spaces as $X^i = \cup X^i_n$ and let $\{K_j\}$ be any countable collection of compact subsets of $\Pi X^i$. Clearly each $K_j$ is contained in some $\Pi_j X^i_{nj}$. Choose some $p \in \Pi X^i$ so that for each i, $p_i \notin X^i_{ni}$. Then p doesn't belong to any $K_j$.

Now for a surprising consequence. In Tychonoff spaces, a non-compact, $\sigma$-compact space isn't pseudocompact and thus contains a closed copy of $\mathbf{N}$. Hence a countable product of such spaces contains a closed copy of $\mathbf{N}^{\mathbf{N}}$ which fails to be $\sigma$-compact by 16). Thus

17) *The product of countably many non-compact,* $\sigma$*-compact, regular spaces is never* $\sigma$*-compact.*

Historically this investigation began with a question about weak topologies in the product of two spaces one of which was $k_\omega$. Specifically, if we cross the rationals with the free group over a compact Hausdorff space does the product have the weak topology of $\mathbf{Q} \times (F_G(K,p))_n$? In general, if Y is some space and $X = \cup X_n$ is a $k_\omega$-space, need $Y \times X$ have the weak topology of the $Y \times X_n$? The answer is no, even if Y is very nice, say the rationals. To see this think of $S_2$ as consisting of a sequence $\{s_i\}$ converging to a point $s_o$ with a sequence of isolated points $\{s_{j,i}\}$ converging to each $s_j$. For each j, let $\{q_{j,i}\}$ be a sequence in $\mathbf{Q}$ converging to $\pi/n$. Let $F = \{(q_{j,i}, s_{j,i})\}$ as j and i both vary. Let

$X_n = \{s_o\} \cup \{s_j | j \in N\} \cup \{s_{j,i} | j \leq n\}$. The $X_n$ thus defined form a $k_\omega$-decomposition of $S_2$. Now for each n, $F \cap (\mathbf{Q} \times X_n)$ is closed in $\mathbf{Q} \times S_2$, but F itself isn't closed since it doesn't contain its limit point $(0,s_o)$. Thus the weak topology from the $\mathbf{Q} \times X_n$ isn't the product topology.

We're now in a position to produce the oft promised demonstration that $\mathbf{Q}$ is not a $k_\omega$-space. Suppose $\mathbf{Q} = \cup Y_n$ were a $k_\omega$-decompositon and further suppose that $F \subseteq \mathbf{Q} \times S_2$ met each $\mathbf{Q} \times X_n$ in a closed set. Then for each n, $F \cap (Y_n \times X_n) = [F \cap (\mathbf{Q} \times X_n)] \cap (Y_n \times X_n)$ must also be closed. Thus F would be closed and the weak topology of the $\mathbf{Q} \times X_n$ would coincide with the product topology, a contradiction.

Of course, what keeps the $\mathbf{Q} \times X_n$ from generating the product topology on $\mathbf{Q} \times S_2$ is $\mathbf{Q}$'s lack of local compactness.

18) *When* Y *is a locally compact Hausdorff space and* $X = \cup Y_n$ *is a* $k_\omega$-*decomposition, the weak topology of the sets* $Y \times X_n$ *coincides with the product topology on* $Y \times X$.

For the proof we exploit the fact that $\beta Y$, being compact, is a $k_\omega$-space with trivial $k_\omega$-decomposition. Since Y is locally compact, it is an *open* subset of $\beta Y$ and thus each $Y \times X_n$ is open in $\beta Y \times X_n$. Suppose $U \subseteq Y \times X$ meets each $Y \times X_n$ in an open set. Then for each n, $U \cap (\beta Y \times X_n)$ is open in $\beta Y \times X_n$ and thus U is open in $\beta Y \times X$. But then U must also be open in $Y \times X$ and 18) is proved.

The local-compactness condition in 18), while sufficient, is by no means necessary. For an example, take the product of any non-locally compact $k_\omega$-space (say $S_2$) and itself.

It's well known that the product of two k-spaces need not be again a k-space although local-compactness of one factor is sufficient. One might hope that one factor being a $k_\omega$-space might also suffice. Unfortunately $\mathbf{Q} \times S_2$ is again a counter example. Take F as above. Since each compact subset of $\mathbf{Q} \times S_2$ is contained in some $\mathbf{Q} \times X_n$, F meets each compact subset in a closed set but is not itself closed. Thus $\mathbf{Q} \times S_2$ is not a k-space.

Finally we devote ourselves to the questions of the metrizability and first countability of $k_\omega$-spaces. Clearly a metrizable $k_\omega$-space, being Lindelöf (by 6)), is separable and second countable. Conversely, a $k_\omega$-space is regular, so second countability suffices.

19) *A* $k_\omega$*-space is metrizable iff it is second countable.*

First countability is <u>not</u> enough. Any compact first countable but not metrizable space, e.g., the unit square with the lexicographic order, is an example.

The close relationship among $k_\omega$-spaces between first countability and local compactness will yield further information about metrizability. From 5) we immediately get

20) *Every first countable* $k_\omega$*-space is locally compact.*

Whence

21) *A metrizable* $k_\omega$*-space is locally compact and hence completely metrizable.*

In particular, we see again that $\mathbf{Q}$ cannot be a $k_\omega$-space. Since $k_\omega$-spaces are composed of compact "pieces" it is natural to ask if the metrizability of the pieces will yield that of the space. In general, the answer is no. For example, the $k_\omega$-decomposition of $S_2$ described in the paragraph following 17) is composed of metrizable pieces while $S_2$ isn't metrizable (in fact isn't first countable or even Fréchet).

For a somewhat simpler example take a sequential fan, i.e., the union of a countable family of convergent sequences with their limit points identified. The pieces are metrizable; the fan is a $k_\omega$-space by 13); however, the fan is not first countable and hence not metrizable.

How about other examples (first countable ones, for instance)? There are none. No essentially different ones at any rate.

22) (Franklin, Thomas [4]) *If* $X = \cup X_n$ *is a* $k_\omega$*-decomposition with each* $X_n$ *metrizable, then* X *is metrizable iff it contains no copy of* $S_2$ *and no sequential fan.*

The proof consists of showing that the hypotheses imply first countability and that first countability implies metrizability. These facts are stated explicitly in the following

23) *If* $X = \cup X_n$ *is a* $k_\omega$*-decomposition with each* $X_n$ *metrizable, then* X *is metrizable iff it is first countable.*

The second, more delicate, lemma is the first countable version of 22) expanded a little.

24) *Suppose* $X = \cup X_n$ *is a* $k_\omega$*-decomposition with each* $X_n$ *first countable and that* X *is not first countable. If* X *is Fréchet it contains a sequential fan. If* X *is not Fréchet, it contains a copy of* $S_2$.

We conclude with an observation and a question concerning $\beta X \setminus X$. An easy corollary of a result due to Fine and Gillman [3] is that the growth of each $k_\omega$-space contains a dense subset which is the union of copies of $\beta\mathbf{N}\setminus\mathbf{N}$. Is it possible that the growth of every $k_\omega$-space (i.e., $\beta X\setminus X$) is an F-space?

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