A SURVEY OF k_ω-SPACES

by

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The k_ω-spaces, a natural generalization of countable
CW-complexes, have appeared recently in papers on topologi-
cal groups and topological semigroups. They seem destined
to play an important role in that study. Here we survey
the purely topological behavior of such spaces. Being unusu-
ally well endowed k-spaces they behave very nicely indeed.
In particular, they possess excellent separation properties,
a nice external characterization, and a very interesting
metrization theorem.

Writing $\mathbb{R} = \cup_{n=1}^{\infty}[-n,n]$ expresses the real line as the
union of an increasing sequence of compact Hausdorff subsets,
with $\mathbb{R}$ having the weak topology with respect to the $[-n,n]$
(i.e., F is closed in $\mathbb{R}$ iff F $\cap$ $[-n,n]$ is closed in $[-n,n]$
for each n). Such a decomposition of a space (i.e., X =
$\cup_{n=1}^{\infty} X_n$ with the X_n compact Hausdorff and increasing, and X
having their weak topology) is called a k_ω-*decomposition*. A
space possessing a k_ω-decomposition is called a k_ω-*space*.
Hence $\mathbb{R}$ is a k_ω-space.

Caution: If we enumerate the rationals $\mathbb{Q} = \{q_1, q_2,$
$q_3, \cdots, q_n, \cdots\}$ and write $\mathbb{Q}_n = \{q_1, q_2, \cdots, q_n\}$ then we may
express $\mathbb{Q} = \cup_{n=1}^{\infty} \mathbb{Q}_n$ as the union of an increasing sequence
of compact Hausdorff subspaces but not produce a k_ω-decomposi-
tion. The weak topology arising from this sequence is
discrete. We shall see later that $\mathbb{Q}$ with its usual topology
admits no k_ω-decomposition.

Worse yet can happen. The space S_2 (the prototype of
sequential but not Fréchet spaces, see [1]) admits two de-
compositions into the union of an increasing sequence of
compact Hausdorff subsets, one of them a k_ω-decomposition,
the other not.

The terminology "k_ω-spaces" seems to be due to E. Michael
[8]; Graev discussed such spaces with reference to topologi-
cal groups in 1948 [5], and Morita introduced them under the
name $\mathcal{G}'$ in 1956 [10]. The related notion of hemicompactness
was introduced by Arens in 1946 [2]. A space, X, is said to
be *hemicompact* if it can be written as the union of countably
many compact subspaces, K_n, with the property that each com-
pact subset of X is contained in a finite union of the K_n's.

Some applications of k_ω-spaces are:

A.   If $X = \cup_{n=1}^{\infty} X_n$ is a k_ω-space then so is the Graev
     free group over X, with k_ω-decomposition $F_G(X,p) =$
     $\cup_{n=1}^{\infty} (F_G(X_{n,p}))_n$ (words of reduced length $\leq$ n and
     letters from X_n) (Ordman [12]).

B.   If G and H are topological groups which are k_ω-
     spaces so also is G ⊔ H (Katz [6], Ordman [13]).

These results are interesting in that they describe
the topological structure of the free group, respectively
coproduct, a problem which is unsolved in general.

C.   X is k_ω if and only if C(X) with its compact open
     topology is completely metrizable (putting together
     results of Warner [15] and Mosiman and Wheeler [11]).

Ordman (in [12]) has collected some known facts about
k_ω-spaces. We state them here, without proof, for the con-
venience of the reader.

1) (Graev) *Every k_ω-space is a Hausdorff k-space.*

2) *If $X = \cup X_n$ is a k_ω-decomposition and f: $X \to Y$ is continuous on each X_n, then f is continuous on X.*

3) (Steenrod) *If $X = \cup X_n$ is a k_ω-decomposition, then each compact subset of X is contained in some X_n. That is, X is hemicompact with respect to its k_ω-decomposition.*

4) (Milnor) *If $X = \cup X_n$ and $Y = \cup Y_n$ are k_ω-decompositions, so is $X \times Y = \cup(X_n \times Y_n)$.*

Thus k_ω-spaces are finitely productive.

The crux of 3) is that if $A \subset X$ meets every $X \backslash X_n$ and if we let $x_n \in A \cap (X \backslash X_n)$ then $U_m = X \backslash \{x_n \mid n \geq m\}$ is an open cover of A with no finite subcover. Number 4) is really a theorem about countable CW-complexes, but the same ideas and techniques work for k_ω-spaces.

There's another known and useful fact about k_ω-spaces due to Ordman [13].

5) *In a k_ω-space, a point without a compact neighborhood cannot support a countable neighborhood base.*

k_ω-spaces can, in fact, boast of much better separation properties than 1) indicates. If $X = \cup X_n$ is a k_ω-decomposition, and E and F are disjoint closed subsets of X, then $E \cap X_1$ and $F \cap X_1$ have X_1 neighborhoods U_1 and V_1 whose closures are disjoint. Let $E_1 = E \cup clU_1$ and $F_1 = F \cup clV_1$. Then $E_1 \cap X_2$ and $F_1 \cap X_2$ have X_2 neighborhoods U_2 and V_2 whose closures are disjoint. Continuing, we produce disjoint neighborhoods $U = \cup U_n$ and $V = \cup V_n$ of E and F. Hence X is normal.

In particular we have,

6) (Morita) *Each k_ω-space, being σ-compact and regular, is Lindelöf, paracompact, etc.*

We conclude immediately that a k_ω-space with any of several weaker compactness properties (pseudo-compact, countably compact) is already compact. k_ω-spaces are very nice indeed.

Fact 3) above (each compact set contained in some member of a k_ω-decomposition) is not only true of k_ω-decompositions, but actually characterizes them for k-spaces. Suppose X = $\cup X_n$ where X is a k-space and the X_n form an increasing sequence of compact Hausdorff subspaces of X such that each compact subspace of X is contained in one of the X_n's. Suppose also that F meets each X_n in a closed set. Let $K \subseteq X$ be compact. Then $K \subseteq X_n$ for some n and $F \cap K = (F \cap \overset{\cdot}{X_n}) \cap K$ is closed, and thus F is closed. Stated formally, we have

7) *An increasing sequence of compact Hausdorff subsets covering a k-space X is a k_ω-decomposition iff each compact subset of X is contained in one of them.*

An immediate consequence is that

7') *The k_ω-spaces are precisely the hemicompact k-spaces.*

In 7) we must assume X to be a k-space. Consider **N** $\cup$ {p} $\subseteq \beta$ **N** . Let K_n = {p,1,2,$\cdots$,n}; every compact subset of **N** $\cup$ {p} is contained in some K_n. Thus N $\cup$ {p} is hemicompact but fails to be k_ω.

Here's another useful fact about k_ω-decompositions.

8) *Suppose* $X = \cup X_n$ *is a* k_ω-*decomposition and* $X = \cup X'_m$
   is another increasing cover by compact subsets. If
   each X_n *is contained in some* X'_m, *then the* X'_m
   also form a k_ω-*decomposition, and conversely.*

To see this, suppose F meets each X'_m in a closed set. Given
n, choose m so that $X_n \subseteq X'_m$. $F \cap X_n = (F \cap X'_m) \cap X_n$ is
closed, and hence F is closed. The converse is immediate
from 3).

   It follows from 8) that

9) *Any subsequence of a* k_ω-*decomposition is again a*
   k_ω-*decomposition.*

   Now that we have gained some familiarity with the work-
ings of k_ω-spaces it's only natural to ask "what are they?".
The obvious conjecture is easily shot down: $\mathbb{Q}$ is a σ-com-
pact k-space which isn't k_ω as we'll show below. "But $\mathbb{Q}$
isn't locally compact," you say. Any discrete space of car-
dinality $> \aleph_0$ is locally compact but not k_ω. "What if we
require both?" Now we're getting somewhere.

10) *A locally compact Hausdorff space is* k_ω *iff it is*
    σ-*compact.*

Suppose X is both σ-compact and locally compact, and let
$X = \cup Y_n$ with each Y_n compact. We'll now define a k_ω-decom-
position by recursion. Let $X_1 = Y_1$. Cover each point of X_1
with a compact neighborhood. Reduce to a finite cover. Union
these together and add Y_2 to form X_2. Thus $X_1 \subseteq X_2$, $Y_2 \subseteq X_2$,
and each point of X_1 has a compact neighborhood in X_2. Con-
tinuing in this way we define the subsequent X_n. Now to show

they form a k_ω-decomposition: suppose F meets each X_n in a

closed set and that $x \notin F$.   This x belongs to some X_n and

hence to int X_{n+1}.   Since $F \cap X_{n+1}$ is compact, there is an

open U containing x and missing $F \cap X_{n+1}$. $U \cap$ int X_{n+1} is

then a neighborhood of x missing F, and F is therefore

closed.

    The proof of 10) yields yet another sufficient descrip-

tion of a k_ω-decomposition as *an increasing sequence of*

compact Hausdorff subsets each contained in the interior of

a subsequent one.

    One might hope to extend the equivalence of σ-compact and

k_ω for locally compact spaces to the larger class of k-spaces

locally compact on a dense subset.   (In the presence of

σ-compactness, these are precisely the Baire spaces.)   Un-

fortunately **Q** , with the dyadic rationals made discrete,

provides the counter example (apply 14).

    On the other hand a k_ω-space need not be a Baire space.

The space S_ω ([1]), being σ-compact and nowhere locally com-

pact, isn't Baire.   However, it is the quotient of a countable

disjoint sum of convergent sequences (and their limits) and

is thus a k_ω-space by 13).

    Note that there are k_ω-spaces (like S_2) which aren't

locally compact, so we haven't yet found them all.   But we're

close--just two easy lemmas.

    Although continuous images of k_ω-spaces need not be k_ω

(map **Q** with the discrete topology onto **Q** with its usual

topology), quotients must be

    11) (Morita) *If $X = UX_n$ is a k_ω-decomposition and* q: X → Y

is a quotient map onto a Hausdorff Y, *then the sets*
$q(X_n)$ *form a* k_ω*-decomposition for* Y.

The proof is straightforward and will be omitted.

12) (Morita) *The disjoint topological sum (coproduct)*
of countably many k_ω*-spaces is again a* k_ω*-space*

Let $X^j = \cup X_n^j$ be a k_ω-decomposition for each j. Then $\oplus_j X^j = \cup_{m=1}^\infty (\cup_{n+j=m} X_n^j)$ is a k_ω-decomposition of the coproduct.

Now given a k_ω-space $X = \cup X_n$ we take the coproduct $\oplus X_n$ of its pieces. The map i: $\oplus X_n \to X$ generated by the inclusions $X_n \to X$ is a quotient map since X has the weak topology of the X_n. Thus *an arbitrary* k_ω*-space can be represented as the quotient of a countable coproduct of compact Hausdorff spaces.* By 11) and 12) all such quotients of countable coproducts are k_ω-spaces. Thus we have our desired external characterization. Remembering that the k-spaces are precisely the quotient of locally compact spaces, and keeping 10), 11), 12) in mind, we may write,

13) *The* k_ω*-spaces are precisely the quotients of σ-compact locally compact spaces.*

Or as Morita stated it

13') *The* k_ω*-spaces are precisely the quotients of locally compact Lindelöf spaces.*

Now, given the compact Hausdorff spaces, we can construct all k_ω-spaces. Nice, but that still leaves us a lot to know about how they behave. Let's first note some hereditary, and map preservation properties and then go on to look more closely at products.

14) *A closed subspace* Y *of a* k_ω-*space* X = $\cup X_n$ *has a*

   k_ω-*decomposition* Y = $\cup(X_n \cap Y)$.

The proof is easy. Clearly being a k_ω-space is not arbitrarily hereditary since $\mathbb{Q} \subseteq \mathbb{R}$.

For inverse preservation by mappings we need to go to k-mappings, i.e., maps with the property that inverse images of compact subsets are compact.

15) *If* Y = $\cup Y_n$ *is a* k_ω-*decomposition,* X *is Hausdorff, and* p: X $\rightarrow$ Y *is an onto k-map, then the sets* $p^{-1}(Y_n)$ *form a* k_ω-*decomposition for* X.

Since Y is a k-space, p is a closed map (and hence perfect). Suppose F meets every $p^{-1}(Y_n)$ in a closed set. Then $p(F) \cap Y_n = p(F \cap p^{-1}(Y_n))$ is closed for each n so that $p(F)$ is closed. Thus $clF \subseteq p^{-1}p(F)$. If $x \in clF\backslash F$, then x is not in the compact set $p^{-1}(p(x)) \cap F$. Thus we may separate these two by disjoint open neighborhoods U and V, i.e., $x \in U$, $p^{-1}(p(x)) \cap F \subseteq V$. Since $x \in clF$, U meets F. Thus $x \in clF\backslash V$. If we now start the argument anew with $F\backslash V$ instead of F we find that $p^{-1}(p(x))$ must meet $F\backslash V$ which is absurd. Thus F is closed and we are done.

On to products. We have seen in 4) that finite products work. One would immediately suppose that uncountable products won't. What about countable products? Assume that $X_1 \subseteq X_2 \subseteq \cdots$ is a sequence of compact subsets of $\mathbb{R}^N$. Then each X_i is contained in some compact box $\Pi_j[-n_i^j, n_i^j]$. But then the compact box $\Pi_j[-n_j^j-1, n_j^j+1]$ is contained in no X_i contradicting 3). Thus $\mathbb{R}^N$ is not a k_ω-space. So, not all countable

products work. Which ones do? Answer: None (except trivial cases).

   16) *The product of countably many non-compact* k_ω*-spaces*
       is never σ*-compact.*

Write the spaces as $X^i = \cup X^i_n$ and let $\{K_j\}$ be any countable collection of compact subsets of ΠX^i. Clearly each K_j is contained in some $\Pi_j X^i_{nj}$. Choose some $p \in \Pi X^i$ so that for each i, $p_i \notin X^i_{ni}$. Then p doesn't belong to any K_j.

   Now for a surprising consequence. In Tychonoff spaces, a non-compact, σ-compact space isn't pseudocompact and thus contains a closed copy of **N**. Hence a countable product of such spaces contains a closed copy of $\mathbf{N}^\mathbf{N}$ which fails to be σ-compact by 16). Thus

   17) *The product of countably many non-compact,* σ*-compact,*
       regular spaces is never σ*-compact.*

   Historically this investigation began with a question about weak topologies in the product of two spaces one of which was k_ω. Specifically, if we cross the rationals with the free group over a compact Hausdorff space does the product have the weak topology of $\mathbf{Q} \times (F_G(K,p))_n$? In general, if Y is some space and $X = \cup X_n$ is a k_ω-space, need Y × X have the weak topology of the Y × X_n? The answer is no, even if Y is very nice, say the rationals. To see this think of S_2 as consisting of a sequence $\{s_i\}$ converging to a point s_o with a sequence of isolated points $\{s_{j,i}\}$ converging to each s_j. For each j, let $\{q_{j,i}\}$ be a sequence in $\mathbf{Q}$ converging to π/n. Let $F = \{(q_{j,i}, s_{j,i})\}$ as j and i both vary. Let

$X_n = \{s_o\} \cup \{s_j \mid j \in N\} \cup \{s_{j,i} \mid j \leq n\}$. The X_n thus defined form a k_ω-decomposition of S_2. Now for each n, $F \cap (\mathbb{Q} \times X_n)$ is closed in $\mathbb{Q} \times S_2$, but F itself isn't closed since it doesn't contain its limit point $(0, s_o)$. Thus the weak topology from the $\mathbb{Q} \times X_n$ isn't the product topology.

We're now in a position to produce the oft promised demonstration that $\mathbb{Q}$ is not a k_ω-space. Suppose $\mathbb{Q} = \cup Y_n$ were a k_ω-decompositon and further suppose that $F \subseteq \mathbb{Q} \times S_2$ met each $\mathbb{Q} \times X_n$ in a closed set. Then for each n, $F \cap (Y_n \times X_n) = [F \cap (\mathbb{Q} \times X_n)] \cap (Y_n \times X_n)$ must also be closed. Thus F would be closed and the weak topology of the $\mathbb{Q} \times X_n$ would coincide with the product topology, a contradiction.

Of course, what keeps the $\mathbb{Q} \times X_n$ from generating the product topology on $\mathbb{Q} \times S_2$ is $\mathbb{Q}$'s lack of local compactness.

18) *When Y is a locally compact Hausdorff space and*
    $X = \cup Y_n$ *is a k_ω-decomposition, the weak topology of*
    the sets $Y \times X_n$ coincides with the product topology
    on $Y \times X$.

For the proof we exploit the fact that βY, being compact, is a k_ω-space with trivial k_ω-decomposition. Since Y is locally compact, it is an *open* subset of βY and thus each $Y \times X_n$ is open in $\beta Y \times X_n$. Suppose $U \subseteq Y \times X$ meets each $Y \times X_n$ in an open set. Then for each n, $U \cap (\beta Y \times X_n)$ is open in $\beta Y \times X_n$ and thus U is open in $\beta Y \times X$. But then U must also be open in $Y \times X$ and 18) is proved.

The local-compactness condition in 18), while sufficient, is by no means necessary. For an example, take the product of any non-locally compact k_ω-space (say S_2) and itself.

It's well known that the product of two k-spaces need
not be again a k-space although local-compactness of one factor
is sufficient. One might hope that one factor being a k_ω-space
might also suffice. Unfortunately $\mathbb{Q} \times S_2$ is again a counter
example. Take F as above. Since each compact subset of $\mathbb{Q} \times S_2$
is contained in some $\mathbb{Q} \times X_n$, F meets each compact subset in
a closed set but is not itself closed. Thus $\mathbb{Q} \times S_2$ is not
a k-space.

Finally we devote ourselves to the questions of the
metrizability and first countability of k_ω-spaces. Clearly
a metrizable k_ω-space, being Lindelöf (by 6)), is separable
and second countable. Conversely, a k_ω-space is regular, so
second countability suffices.

19) *A k_ω-space is metrizable iff it is second countable.*

First countability is <u>not</u> enough. Any compact first
countable but not metrizable space, e.g., the unit square
with the lexicographic order, is an example.

The close relationship among k_ω-spaces between first
countability and local compactness will yield further in-
formation about metrizability. From 5) we immediately get

20) *Every first countable k_ω-space is locally compact.*

Whence

21) *A metrizable k_ω-space is locally compact and hence*
    completely metrizable.

In particular, we see again that $\mathbb{Q}$ cannot be a k_ω-space.
Since k_ω-spaces are composed of compact "pieces" it is

natural to ask if the metrizability of the pieces will yield
that of the space. In general, the answer is no. For example,
the k_ω-decomposition of S_2 described in the paragraph follow-
ing 17) is composed of metrizable pieces while S_2 isn't
metrizable (in fact isn't first countable or even Fréchet).

For a somewhat simpler example take a sequential fan,
i.e., the union of a countable family of convergent sequences
with their limit points identified. The pieces are metriza-
ble; the fan is a k_ω-space by 13); however, the fan is not
first countable and hence not metrizable.

How about other examples (first countable ones, for in-
stance)? There are none. No essentially different ones at
any rate.

22) (Franklin, Thomas [4]) *If* X = $\bigcup X_n$ *is a* k_ω-*decomposi-*
tion with each X_n *metrizable, then* X *is metrizable*
iff it contains no copy of S_2 *and no sequential fan.*
The proof consists of showing that the hypotheses imply first
countability and that first countability implies metrizability.
These facts are stated explicitly in the following

23) *If* X = $\bigcup X_n$ *is a* k_ω-*decomposition with each* X_n *metriza-*
ble, then X *is metrizable iff it is first countable.*

The second, more delicate, lemma is the first countable
version of 22) expanded a little.

24) *Suppose* X = $\bigcup X_n$ *is a* k_ω-*deocmposition with each* X_n
first countable and that X *is not first countable.*
If X *is Fréchet it contains a sequential fan. If* X
is not Fréchet, it contains a copy of S_2.

We conclude with an observation and a question concern-
ing $\beta X \backslash X$.   An easy corollary of a result due to Fine and
Gillman [3] is that the growth of each k_ω-space contains a
dense subset which is the union of copies of $\beta \mathbf{N} \backslash \mathbf{N}$.   Is it
possible that the growth of every k_ω-space (i.e., $\beta X \backslash X$) is an
F-space?

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