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## REFINABLE MAPS ON GRAPHS ARE NEAR HOMEOMORPHISMS<sup>1</sup>

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In recent papers J. W. Rogers, Jr., Jo Ford and others have studied refinable maps, i.e., continuous functions on compacta that can be approximated arbitrarily closely by onto  $\epsilon$ -maps, for any positive number  $\epsilon$ . In this paper it is shown that if  $r$  is a refinable map defined on a (finite) graph  $X$ , then  $r(X)$  is a homeomorphic to  $X$  and, in fact,  $r$  is a near homeomorphism, i.e., can be approximated arbitrarily closely by homeomorphisms. A stronger condition on  $r$ , with the graph hypothesis shifted to the image, also yields the same result. It is also shown that, while the image of a regular curve under a refinable map is a regular curve (such refinable maps are monotone), it need not be homeomorphic to the domain.

### 1. Introduction and Basic Definitions

All spaces considered here are compact and metric, and all maps are continuous functions.

*Definitions.* If  $\epsilon$  is a positive number and  $f$  maps  $X$  onto  $Y$ , then  $f$  is an  $\epsilon$ -map if the diameter of  $f^{-1}(y)$  is less than  $\epsilon$  for each  $y$  in  $Y$ . If each of  $r$  and  $f$  is a map from  $X$  onto  $Y$ , then  $f$  is an  $\epsilon$ -refinement of  $r$  if  $f$  is an  $\epsilon$ -map and  $d(f(x), r(x)) < \epsilon$ , for each  $x$  in  $X$ . A map  $r$  is (monotonely)

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<sup>1</sup>This paper is an extension of some of the work presented to the 1977 Spring Topology Conference at L.S.U. March 10, 1977, under the title *Refinable functions on finite graphs and on spaces having the proximate f.p.p.*

*refinable* or (*monotonely*) *1-refinable* if, for each positive number  $\epsilon$ , there is an  $\epsilon$ -refinement of  $r$  (that is monotone). A map  $r$  is (*monotonely*) *2-refinable* if, for each positive number  $\epsilon$ , there is a refinable map that is an  $\epsilon$ -refinement of  $r$  (that is monotone).

Any refinable map onto a locally connected space is monotone [1, Corollary 1.2], so any 2-refinable map onto a locally connected space is monotonely refinable. The reverse is not true.

*Definitions.* A *graph* is a compact metric space that is the union of a finite, disjoint collection of sets each of which is either a singleton set or an open, connected subset of an arc in the space. A connected space is a *regular curve* if it has a basis consisting of sets with finite boundary.

The following results are presented herein. (1) A refinable map defined on a graph is a near homeomorphism. (2) A monotonically refinable map onto a graph is a near homeomorphism. (3) There is a refinable map from the Warsaw Circle onto a simple closed curve. (4) There is a monotonely refinable map (that is not 2-refinable) that maps a regular curve onto a regular curve that is not homeomorphic to the domain.

$d$  is the distance function in any space and, also,  $d(A)$  is the diameter of any point set  $A$ . It will be clear from context what use of  $d$  is intended.

## 2. Graph Domains

A result of Jo Ford and J. W. Rogers, Jr. [1, Corollary 3.2] concerning refinable maps on ANR's and a result of Jack

Segal [4, Theorem 2.2, p. 359] concerning quasi dimension type (introduced in [3]) are used to show that the image of a graph under a refinable map is homeomorphic to the domain. The fact that a monotone map from an arc onto an arc is a near homeomorphism is then used to show that a refinable map on a graph is a near homeomorphism.

In this context the relevant definition concerning quasi dimension type simplifies to the following.

*Definition.* We write  $qX \leq qY$  and say  $X$  is  $Y$ -like if, for each positive number  $\epsilon$ , there is an  $\epsilon$ -map of  $X$  onto  $Y$ . We write  $qX = qY$  and say  $X$  and  $Y$  are *quasi-homeomorphic* if  $qX \leq qY$  and  $qY \leq qX$ .

*Theorem 1.* If  $X$  is a graph and  $r: X \rightarrow Y$  is a refinable map onto  $Y$ , then  $X$  and  $Y$  are homeomorphic.

*Proof.* By [1, Corollary 3.2], the domain and range of a refinable map are quasi-homeomorphic if the domain is an ANR. Since any graph is an ANR, and  $X$  has a finite number of components,  $Y$  has the same number of components. Since  $r$  is continuous,  $r$  maps components of  $X$  into components of  $Y$ . It follows that  $r$  maps different components of  $X$  onto different components of  $f(X)$ . Let  $C$  be any component of  $X$  and let  $\epsilon = d(r(C), r(X) \setminus r(C))$ . If  $f$  is an  $\epsilon$ -refinement of  $r$  then  $f|C$  is an  $\epsilon$ -refinement of  $r|C$ . It follows that  $r|C$  is refinable. Therefore it is sufficient to prove the theorem for the case where  $X$  is connected. By [1, Corollary 3.2] the ANR  $X$  is quasi-homeomorphic to  $Y$ , i.e.  $qX = qY$ . In addition, since  $X$  is locally connected  $Y$  is also. But any refinable map onto a locally connected space is monotone [1, Corollary

1.2] and any monotone image of a graph is a graph, so  $Y$  is a graph. Finally, by [4, Theorem 2.2, p. 359], the connected graphs  $X$  and  $Y$  are homeomorphic since  $qX = qY$ .

*Theorem 2.* If  $X$  is a graph and  $f: X \rightarrow Y$  is a monotone map onto  $Y$  such that for each  $y$  in  $Y$ , the inverse set  $f^{-1}(y)$  contains no simple closed curve and contains at most one point not of order two, then  $f$  is a near homeomorphism.

*Proof.* Let  $T$  be the set of all points in  $X$  of order two. Let  $\{A_1, \dots, A_n\}$  be the set of components of  $T$ . For  $i$  in  $\{1, \dots, n\}$ , let  $a_i$  be in  $A_i \setminus f^{-1}(f(\bar{A}_i \setminus A_i))$ . Such a point exists since  $f(\bar{A}_i) \neq f(\bar{A}_i \setminus A_i)$ , for all  $i$ . Let  $\{B_1, \dots, B_{2n}\}$  be the set of all components of  $T \setminus \{a_1, \dots, a_n\}$ . Whereas some  $\bar{A}_i$  might be a simple closed curve, each  $\bar{B}_j$  is an arc. Let  $b_j$  and  $c_j$  be the end points of  $\bar{B}_j$ . For  $j$  in  $\{1, \dots, 2n\}$ , the function  $f|_{\bar{B}_j}$  is a nonconstant, monotone map defined on an arc and, hence, is a near homeomorphism. Let  $\epsilon$  be a positive number less than  $\min \{d(f(b_j), f(c_j)) \mid j \in \{1, \dots, 2n\}\}$  and, for each  $j$ , let  $h_j$  be a homeomorphism from  $\bar{B}_j$  onto  $f(\bar{B}_j)$  such that  $d(h_j(b), f(b)) < \epsilon$ , for each  $b$  in  $\bar{B}_j$ . Let  $h = \bigcup_{j=1}^{2n} h_j$ . Then  $h$  is a homeomorphism from  $X$  onto  $Y$  such that  $d(h(x), f(x)) < \epsilon$ , for each  $x$  in  $X$ . Hence  $f$  is a near homeomorphism.

*Corollary 1.* If  $X$  is a graph and  $r: X \rightarrow Y$  is a refinable map onto  $Y$ , then  $r$  is a near homeomorphism.

*Proof.* By Theorem 1, the space  $Y$  is homeomorphic to  $X$ . But that would not be the case if, for some  $y$  in  $Y$ , the set  $r^{-1}(y)$  either contained two points not of order two or contained a simple closed curve. Hence, by Theorem 2, the map  $r$  is a near homeomorphism.

### 3. Graph Images

A result of Sibe Mardešić and Jack Segal [2, Th. 3, p. 160] concerning locally connected,  $\Pi$ -like continua (where  $\Pi$  is a class of polyhedra) and 0-regular convergence is used, with results concerning 0-regular convergence of arcs and simple closed curves [5, p. 432], to prove a theorem that yields the corollary that a locally connected continuum that can be mapped onto a graph by a refinable map is homeomorphic to the graph. Two lemmas are then presented that allow the local connectedness condition on the domain to be replaced by the condition that the map be monotonely refinable.

*Definitions.* If  $\Pi$  is a class of polyhedra and  $X$  is a compactum, then  $X$  is (monotonely)  $\Pi$ -like if, for every positive number  $\varepsilon$ , there is an  $\varepsilon$ -map (that is monotone) from  $X$  onto some member of  $\Pi$ . A sequence  $A_1, A_2, \dots$  of closed sets is said to converge 0-regularly [6] to a limit set  $A$ , if it converges to  $A$  and, for each positive number  $\varepsilon$ , there are positive numbers  $\delta$  and  $N$  such that, for any integer  $n > N$ , any points  $p$  and  $q$  in  $A_n$ , such that  $d(p, q) < \delta$ , lie in a connected subset  $B$  of  $A_n$  with  $d(B) < \varepsilon$ .

*Theorem 3.* If  $Y$  is a graph and  $X$  is a locally connected  $\{Y\}$ -like compactum, then  $X$  is a graph that is homeomorphic to a monotone decomposition of  $Y$ .

*Proof.* Any continuous function maps any component of  $X$  into some component of  $Y$ . Also, for any two components of  $X$ , there is some  $\varepsilon$ -map that maps them into different components of  $Y$ . It follows that, for some positive number  $\varepsilon$ , any  $\varepsilon$ -map from  $X$  onto  $Y$  yields a one-to-one correspondence

between the components of  $X$  and the components of  $Y$  such that for any component  $K$  of  $Y$  the corresponding component of  $X$  is  $\{K\}$ -like. Consequently, we can assume without loss of generality that  $X$  and  $Y$  are connected. Since the conclusion of the theorem is obvious if  $X$  is degenerate we will assume  $X$  is nondegenerate and hence, since  $X$  is  $Y$ -like, that  $Y$  is nondegenerate.

By [2, Th. 3, p. 160], there is a sequence  $P_1, P_2, \dots$ , of homeomorphic images of  $Y$ , that converges 0-regularly to  $X$ . For each natural number  $i$ , let  $h_i$  be a homeomorphism from  $Y$  onto  $P_i$ . Now  $Y$  is a nondegenerate connected graph so  $Y = \bigcup_{i=1}^n \bar{A}_i$ , where each  $A_i$  is an open arc, i.e., an open set that is homeomorphic to an open interval of real numbers, such that  $\bar{A}_i$  is an arc. For  $i = 1, \dots, n$ , let  $b_i$  and  $c_i$  be the end points of  $\bar{A}_i$ .

Since any subsequence of a sequence that converges 0-regularly also converges 0-regularly to the same set, we can assume without loss of generality that  $h_1(b_i), h_2(b_i), \dots; h_1(c_i), h_2(c_i), \dots; and  $h_1(\bar{A}_i), h_2(\bar{A}_i), \dots$  all converge, for  $i = 1, \dots, n$ .$

If  $h_1(\bar{A}_i), h_2(\bar{A}_i), \dots$  does not converge to a point, then, for  $k = 1, 2, \dots$ , there is a point  $t_k$  in  $h_k(A_i)$  such that  $t_1, t_2, \dots$  converges to a point not in  $\lim_{k \rightarrow \infty} h_k(\{b_i, c_i\})$ . Without loss of generality we can assume that the homeomorphisms are defined so that, there is a point  $a_i$  in  $A_i$  such that  $t_k = h_k(a_i)$ , for  $k = 1, 2, \dots$ . Let  $B_i$  and  $C_i$  be the components of  $A_i \setminus \{a_i\}$  with end points  $b_i$  and  $c_i$ , respectively. Since  $\lim_{k \rightarrow \infty} h_k(a_i) \neq \lim_{k \rightarrow \infty} h_k(b_i)$ , and  $P_1, P_2, \dots$  converges 0-regularly,  $h_1(\bar{B}_i), h_2(\bar{B}_i), \dots$  converges 0-regularly and does

not converge to a point. By [5, p. 432],  $\lim_{k \rightarrow \infty} h_k(\bar{B}_1)$  is an arc. It follows from the 0-regularity of the convergence of  $P_1, P_2, \dots$  and the openness of  $B_1$ , that the boundary of  $\lim_{k \rightarrow \infty} h_k(\bar{B}_1)$  is its set of end points. Similarly  $\lim_{k \rightarrow \infty} h_k(\bar{C}_1)$  is an arc whose boundary is its set of end points.

If  $\lim_{k \rightarrow \infty} h_k(\bar{A}_i)$  is nondegenerate for each  $i$ , then  $X$  and  $Y$  are unions of the collections of arcs

$$\{\lim_{k \rightarrow \infty} h_k(\bar{B}_1), \dots, \lim_{k \rightarrow \infty} h_k(\bar{B}_n), \lim_{k \rightarrow \infty} h_k(\bar{C}_1), \dots, \lim_{k \rightarrow \infty} h_k(\bar{C}_n)\}$$

and  $\{\bar{B}_1, \dots, \bar{B}_n, \bar{C}_1, \dots, \bar{C}_n\}$ , respectively, which are "stuck together" in the same way. Hence  $X$  and  $Y$  are then homeomorphic.

If, on the other hand,  $\lim_{k \rightarrow \infty} h_k(\bar{A}_i)$  is degenerate, for some  $i$ , then we can assume without loss of generality that there is an  $m$  in  $\{1, \dots, n-1\}$  such that  $\lim_{k \rightarrow \infty} h_k(\bar{A}_i)$  is a point if and only if  $i$  is in  $\{1, \dots, m\}$ . Let  $Y_1$  be the decomposition space of  $Y$  whose only nondegenerate element is  $\bar{A}_1$  and let  $f_1$  be the projection map from  $Y$  onto  $Y_1$ . For  $j = 1, \dots, m-1$ , let  $Y_{j+1}$  be the decomposition space of  $Y_j$  whose only nondegenerate element is  $[f_j \circ \dots \circ f_1](\bar{A}_{j+1})$ . Let  $f_{j+1}$  be the projection map from  $Y_j$  onto  $Y_{j+1}$ . Now  $Y_m$  and  $X$  both consist of  $2(n-m)$  arcs "stuck together" in the same way and, hence, are homeomorphic. Also  $Y_m$  is the image of  $Y$  under the monotone map  $f_m \circ \dots \circ f_1$ . Hence  $X$  is homeomorphic to a monotone decomposition of  $Y$  and therefore is a graph.

*Corollary 2. If  $X$  is locally connected,  $Y$  is a graph,*

and there is a refinable map  $r: X \rightarrow Y$  onto  $Y$  then  $X$  and  $Y$  are homeomorphic and  $r$  is a near homeomorphism.

*Proof.* By the definition of refinability,  $X$  is  $\{Y\}$ -like. Hence by Theorem 3,  $X$  is a graph. It then follows from Theorem 1 that  $X$  and  $Y$  are homeomorphic and, by Corollary 1, that  $r$  is a near homeomorphism.

We now present two lemmas that permit us to alter Corollary 2 by replacing the condition that  $X$  is locally connected by the condition that  $r$  is monotonely refinable.

*Lemma 1.* If  $\Pi$  is a class of locally connected compacta and  $X$  is monotonely  $\Pi$ -like, then  $X$  is locally connected.

*Proof.* Let  $x \in X$  and let  $\epsilon$  be a positive number. Let  $f$  be a monotone  $\epsilon/2$ -map onto some member  $Y$  of  $\Pi$ . There is a positive number  $\delta$  such that if  $f(x) \in A$  and  $d(A) < \delta$  then  $d(f^{-1}(A)) < \epsilon$ . Since  $Y$  is locally connected, there is a connected open set  $A$  in  $Y$  such that  $f(x) \in A$  and  $d(A) < \delta$ . Then  $f^{-1}(A)$  is a connected open set of diameter less than  $\epsilon$  containing  $x$ . It follows that  $X$  is locally connected.

*Lemma 2.* If  $r: X \rightarrow Y$  is a monotonely refinable map onto  $Y$  and  $Y$  is locally connected then  $X$  is monotonely  $\{Y\}$ -like and, hence, locally connected.

*Corollary 3.* If  $r: X \rightarrow Y$  is a monotonely refinable map onto the graph  $Y$ , then  $X$  and  $Y$  are homeomorphic and  $r$  is a near homeomorphism.

*Corollary 4.* If  $r: X \rightarrow Y$  is a 2-refinable map onto the graph  $Y$ , then  $X$  and  $Y$  are homeomorphic.

**4. Examples and Question**

The fact that neither Theorem 1 nor Corollary 3 can be generalized to apply to all regular curves is established by the following example.

*Example 1.* A monotonely refinable map (that is not 2-refinable) from a regular curve  $X$  onto a topologically different regular curve  $Y$ .

Let  $N(1), N(2), \dots$  be a sequence of positive integers such that, for each positive integer  $n$ , the set  $N^{-1}(n)$  is infinite. The domain of the monotonely refinable map is the union of continua  $T_0, T_1, \dots$ , that can be described in terms of polar coordinates as follows.

$$T_0 = \{(\rho, \theta) \mid 0 \leq \rho \leq 1\} \cup \left[ \bigcup_{j=1}^{\infty} \{(\rho, \theta) \mid \rho = 1/2^j \text{ and } -1/2^j \leq \theta \leq 0\} \right].$$

For each positive integer  $i$ , the continuum  $T_i$  is the union of  $\{(\rho, 1/2^i) \mid 0 \leq \rho \leq 1/2^i\}$  and  $N(i)$  "stickers" of the form  $\{(1/2^j, \theta) \mid 1/2^i \leq \theta \leq 1/2^i + 1/2^j\}$  where  $j > i$ .

The image set  $Y$  is  $\bigcup_{i=1}^{\infty} T_i$  and the monotonely refinable map  $r$  from  $X$  onto  $Y$  is the identity on  $Y \subset X$  and maps  $T_0$  to  $(0,0)$ .

To see that  $r$  is monotonely refinable, let  $\epsilon$  be any positive number, let  $n$  be the number of "stickers" of diameter greater than or equal to  $\epsilon$  on  $T_0$ , and let  $T_{M(1)}, T_{M(2)}, \dots$  be a subsequence of  $T_1, T_2, \dots$  such that, for  $i$  any positive integer,  $d(T_{M(i)}) < \epsilon/2$  and the number of stickers on  $T_{M(i)}$  is  $n$ . Let  $f: X \rightarrow Y$  be the identity on  $Y \setminus [T_0 \cup (\bigcup_{i=1}^{\infty} T_{M(i)})]$ . For each positive integer  $i$ , let  $f$  map  $T_{M(i)}$  homeomorphically onto  $T_{M(i+1)}$ , leaving  $(0,0)$  fixed. Let  $f$  map  $T_0$  onto  $T_{M(1)}$

monotonely so that  $f$  is one-to-one on  $T_0$  except on the "stickers" of diameter less than  $\epsilon$ . The map  $f$  is a monotone  $\epsilon$ -refinement of  $r$ .

It is clear that  $X$  and  $Y$  are regular curves that are not homeomorphic.

To see that  $r$  is not 2-refinable, one shows that  $f$  is essentially typical of the monotone  $\epsilon$ -refinement of  $r$ , in the way it maps  $T_0$  into  $Y$ , and that such maps are not refinable.

Note that  $X$ , in Example 1, does not have finite order, i.e., is not of order  $n$  or less at each point, for any positive number  $n$ . There is a similar example of a refinable (but not monotonely refinable) map from a regular curve of order 3 onto a topologically different regular curve of order 3. The domain and image sets are the union of  $\{(x,y) \mid x = 0 \text{ and } 0 \leq y \leq 1/2\}$  and the sets one gets as  $X$  and  $Y$  respectively, in Example 1, when  $(a,b)$  in the description is the point with rectangular (rather than polar) coordinates  $a$  and  $b$ . The refinable map is similar to  $r$ , especially on the  $T_i$ 's.

*Question.* Is there a monotonely refinable map from a regular curve of finite order onto a topologically different regular curve of finite order?

The following example, which was called to my attention by Jack Rogers, shows that the condition that  $r$  be monotonely refinable, rather than merely refinable, in Corollary 3 is necessary.

*Example 2.* A refinable map from the Warsaw Circle onto a simple closed curve.

The projection map from the Warsaw Circle onto the

simple closed curve resulting from shrinking the limiting bar to a point is refinable (but, of course, not monotonely refinable).

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