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Introduction

Let $\mathcal{B}$ be a base for a topological space X. $\mathcal{B}$ is said to be an *ortho-base* if whenever $\mathcal{B}' \subset \mathcal{B}$ and $p \in \bigcap\mathcal{B}'$, then either $\cap\mathcal{B}'$ is open, or $\mathcal{B}'$ contains a local base at p. $\mathcal{B}$ is said to have *subinfinite rank* (*rank n*) if whenever $\mathcal{B}' \subset \mathcal{B}$, $\cap\mathcal{B}' \neq \emptyset$, and $\mathcal{B}'$ is infinite ($\mathcal{B}'$ has cardinality n + 1), then at least two elements of $\mathcal{B}'$ are related by set inclusion. Spaces having an ortho-base, and spaces having a base of subinfinite rank both generalize two more familiar classes of spaces, namely the non-archimedean spaces, and the spaces having a uniform base. These concepts were introduced by P. J. Nyikos [6], and studied in depth by W. F. Lindgren and Nyikos [5]. The spaces having a base of subinfinite rank were also studied by Nyikos and the author in [3].

It is the purpose of this paper to answer a number of questions of Nyikos concerning these two classes of spaces. Among other results, we prove that countably compact spaces with a base of either type are compact, separable spaces with a base of subinfinite rank are hereditarily Lindelöf, separable spaces with a base of point-finite rank are metrizable, and monotonically normal spaces with an ortho-base are paracompact.

Main Results

Our first few results lead up to the proof that countably compact spaces with an ortho-base or a base of subinfinite rank are compact, but some are also interesting by themselves. The first two lemmas reduce the problem to the separable case, and then the first two theorems finish it off. Lemma 1 was proved independently by Dennis Burke.

Lemma 1. *Let* X *be countably compact but not compact. Suppose every separable closed subspace of* X *is compact. Then* X *contains a perfect pre-image of* ω_1.

Proof. X is not Lindelöf, so there exists an open cover $\mathcal{U}$ of X with no countable subcover. Let $\mathcal{U}_0 = \{\emptyset\}$. Pick $U_0 \in \mathcal{U}$ and $x_0 \in U_0$. Let $\mathcal{U}_1 = \{U_0\}$. Suppose x_α and $\mathcal{U}_\alpha$ have been chosen for all $\alpha < \beta < \omega_1$. Let $X_\beta = \overline{\{x_\alpha\}_{\alpha<\beta}}$, and let $\mathcal{U}_\beta \subset \mathcal{U}$ be a finite cover of X_β. Pick $x_\beta \in X - \cup\{\cup\mathcal{U}_\alpha | \alpha < \beta\}$. Clearly, $\{x_\beta\}_{\beta<\omega_1}$, is a discrete set of points.

For each limit ordinal $\beta < \omega_1$, let Z_β be the set of limit points of $\{x_\alpha\}_{\alpha<\beta}$ which are not limit points of $\{x_\alpha\}_{\alpha<\gamma}$ if $\gamma < \beta$. Clearly Z_β is closed, compact, and non-empty.

For $n \in \omega$, let $Z_n = \{x_n\}$. For $\omega < \alpha < \omega_1$, α not a limit ordinal, let $Z_\alpha = \{x_{\alpha-1}\}$. Let $Z = \cup_{\alpha<\omega_1} Z_\alpha$. Define $f: Z \to \omega_1$ by $f(z) = \alpha$ if and only if $z \in Z_\alpha$.

Clearly $f^{-1}(\alpha)$ is compact for all $\alpha < \omega_1$. Also, $f^{-1}((\alpha,\beta]) = \cup_{\gamma\leq\beta} Z_\gamma - \cup_{\gamma\leq\alpha} Z_\gamma$, which is clopen in Z. Thus f is continuous.

It remains to prove that f is closed. Let A be a closed subset of Z, and suppose f(A) is not closed in ω_1. Let $\beta \in \overline{f(A)} - f(A)$. There exist $\alpha_n \to \beta$ in ω_1 such that $\alpha_n \in f(A)$ for all $n \in \omega$. Let $a_n \in Z_{\alpha_n} \cap A$. The set $\{a_0, a_1, a_2, \cdots\}$ must have a cluster point $z \in X$. It is easy to see that $z \in Z$, hence $z \in A$ and $f(z) = \beta \in f(A)$, contradiction.

Lemma 2. *No perfect pre-image of* ω_1 *has a base of subinfinite rank or an ortho-base.*

Proof. Let $f: X \to \omega_1$ be a perfect map and let $\mathcal{B}$ be a base for X. We shall show that $\mathcal{B}$ is not an ortho-base and does not have subinfinite rank. For each limit ordinal $\alpha < \omega_1$, let $\mathcal{B}(\alpha,0)$ be a finite minimal cover of $f^{-1}(\alpha)$ by elements of $\mathcal{B}$, such that $(\cup\mathcal{B}(\alpha,0)) \cap f^{-1}((\alpha,\omega_1)) = \emptyset$. Let $\beta(\alpha,0)$ be the least ordinal γ such that $f^{-1}((\gamma,\alpha]) \subset \cup\mathcal{B}(\alpha,0)$, and let $\beta'(\alpha,0)$ be the least ordinal δ such that $f^{-1}([0,\delta]) \cap B \neq \emptyset$ whenever $B \in \mathcal{B}(\alpha,0)$ and $B \not\subset f^{-1}(\alpha)$.

The functions $\alpha \to \beta(\alpha,0)$ and $\alpha \to \beta'(\alpha,0)$ are regressive, so there exists an uncountable subset A_0 of ω_1 and $\alpha_0,\alpha_0' \in \omega_1$ such that $\beta(\alpha,0) = \alpha_0$ and $\beta'(\alpha,0) = \alpha_0'$ for all $\alpha \in A_0$. Suppose α_i,α_i', and A_i have been defined for all $i < n$, where $n \in \omega$. Define α_n,α_n', and A_n similar to the case $n = 0$, but if $\alpha > \beta = \max\{\alpha_{n-1},\alpha_{n-1}'\}$, let $B \in \mathcal{B}(\alpha,n)$ imply $B \subset f^{-1}((\beta+1,\alpha])$. Let $\delta = \sup\{\alpha_n | n \in \omega\}$. Note that $\delta = \sup\{\alpha_n' | n \in \omega\}$ also. Let $\{\gamma_n\}_{n\in\omega}$ be an increasing sequence of ordinals larger than δ such that $\gamma_n \in A_n$. Let x be a point of the boundary of $f^{-1}(\delta)$. We see from the definition of α_n that there exists $B_n \in \mathcal{B}(\gamma_n,n)$, for each $n \in \omega$. Let $\mathcal{B}' = \{B_0, B_1,\cdots\}$. It is easy to see from the relations $f^{-1}(\gamma_i) \cap B_i \neq \emptyset$, $f^{-1}([0,\alpha_i]) \cap B_i \neq \emptyset$, $f^{-1}(\gamma_{i+1}) \cap B_i = \emptyset$ and $f^{-1}([0,\alpha_{i-1}]) \cap B_i = \emptyset$ that (i) no two elements of $\mathcal{B}'$ are related, and (ii) $\cap\mathcal{B}' \subset f^{-1}(\delta)$. Property (i) says that $\mathcal{B}$ does not have subinfinite rank and (ii) says that $\mathcal{B}$ is not an ortho-base.

Theorem 1. *Every separable space with an ortho-base has a* G_δ*-diagonal.*

Proof. Let $Q = \{q_1, q_2, \cdots\}$ be a countable dense subset of a space X with ortho-base $\mathcal{B}$. For each $n \in N$, let $\{B_{n,i}\}_{i=1}^{n} \subset \mathcal{B}$ be such that $q_i \in B_{n,i} - \bigcup_{j \neq i} B_{n,j}$. If $x \in X$, and $x \neq q_i$ if $i \leq n$, let $C_{x,n} \in \mathcal{B}$ be such that $x \in C_{x,n}$ and $C_{x,n} \cap \{q_1, \cdots, q_n\} = \emptyset$. Let $\mathcal{U}_n = \{B_{n,i}\}_{i=1}^{n} \cup \{C_{x,n} | x \in X - \{q_1, \cdots, q_n\}\}$.

Pick $y \in X$, and suppose $z \in \bigcap_{n=1}^{\infty} (y, \mathcal{U}_n)$, $z \neq y$. Then there exists $U_n \in \mathcal{U}_n$ such that $\{y,z\} \subset U_n$. Thus $U = \bigcap_{n=1} U_n$ is open, and so there exists $q_k \in Q \cap U$. Therefore, if $n \geq k$, then $U_n = B_{n,k}$. But $(\bigcap_{n=k}^{\infty} B_{n,k}) \cap Q = \{q_k\}$, contradicting the fact that $\bigcap_{n=k}^{\infty} B_{n,k}$ is a non-degenerate open set. Thus $\bigcap_{n=1}^{\infty} st(y, \mathcal{U}_n) = \{y\}$ for every $y \in X$, and so X has a G_δ-diagonal.

In our next theorem, we use the following notation: d(X) is the density of X, hd(X) is the hereditary density (i.e., the supremum of the densities of the subspaces of X), hL(X) is the hereditary Lindelöf degree, and s(X) is the spread (i.e., the supremum of the cardinalities of the discrete subspaces of X). We also use the following theorem of partition calculus, denoted by $\alpha \to (\alpha, \omega)^2$: If the unordered pairs of a set A of cardinality α are put into two sets, set I and set II, then either there exists an infinite subset of A all of whose pairs belong to set II, or there exists a subset of A of cardinality α, all of whose pairs belong to set I.

Theorem 2. *Let* X *be a regular space with a base of* *subinfinite rank. Then* $d(X) = hd(X) \geq hL(X) = s(X)$. *Thus a regular separable space with a base of subinfinite rank is hereditarily separable and hereditarily Lindelöf.*

Proof. We first prove that $d(X) = hd(X)$. Let $\tau = d(X)$, and suppose $hd(X) > \tau$. Then there exists a sequence $\{x_\alpha\}_{\alpha<\tau^+}$ such that $x_\beta \notin \overline{\{x_\alpha\}_{\alpha<\beta}}$ whenever $\beta < \tau^+$. Let $\mathcal{B}$ be a base of subinfinite rank, and for each $\beta < \tau^+$, let $U_\beta \in \mathcal{B}$ be such that $x_\beta \in U_\beta$ and $\overline{U_\beta} \cap \{x_\alpha\}_{\alpha<\beta} = \emptyset$. Since $d(X) = \tau$, there is a fixed subcollection A of $\{U_\alpha | \alpha < \tau^+\}$ such that $|A| = \tau^+$. Applying $\tau^+ \to (\tau^+,\omega)^2$, there exists a subset A' of A of cardinality τ^+ such that $C = \{U_\alpha | U_\alpha \in A'\}$ is a chain. Note that if $\alpha < \alpha'$ and $U_\alpha, U_{\alpha'} \in C$, then $U_\alpha \supset U_{\alpha'}$ and $x_\alpha \in U_\alpha - \overline{U_{\alpha'}}$. For $\alpha < \tau^+$, let $\hat{\alpha} = \min\{\gamma | \gamma > \alpha \text{ and } U_\gamma \in C\}$. Then $\{U_\alpha - \overline{U_{\hat{\alpha}}} | U_\alpha \in C\}$ is a disjoint collection of cardinality τ^+ of open subsets of X, contradiction.

To prove the remaining parts of the theorem, we shall use the following: (*) if $\mathcal{U}$ is a collection of open subsets of X of subinfinite rank, then for each $x \in X$ there exists a subcollection $\mathcal{U}' \subset \mathcal{U}$ such that $|\mathcal{U}'| \leq c(X)$ and $st(x,\mathcal{U}) \subseteq \{\overline{U} | U \in \mathcal{U}'\}$.

Assuming (*) holds, it is easy to see that $d(X) \geq hL(X)$. To see that (*) implies $s(X) \geq hL(X)$, and thus $s(X) = hL(X)$, let Y be any open subset of X and $\mathcal{V}$ an open cover of Y. Let $\mathcal{U}$ be a cover of Y of subinfinite rank such that $U \in \mathcal{U}$ implies $\overline{U} \subset V$ for some $V \in \mathcal{V}$. Pick $y_0 \in Y$. If y_α has been chosen for all $\alpha < \beta$, pick $y_\beta \in Y - \cup_{\alpha<\beta} st(y_\alpha,\mathcal{U})$. Then the y_α's are discrete, thus there are no more than $s(X)$ of them. Applying (*), we get a subcover of Y of cardinality not greater than $s(X)$, hence $hL(X) \leq s(X)$.

It remains to prove that (*) holds. Let $\mathcal{U}$ be a collection of open subsets of X of subinfinite rank, and let $x \in X$. Let $x \in U_0 \in \mathcal{U}$. If U_α has been chosen for all $\alpha < \beta$, let $U_\beta \in \mathcal{U}$ be such that $x \in U_\beta$ and whenever $\alpha < \beta$, $U \not\subset \overline{U_\alpha}$, providing such a set exists. Suppose we have chosen such U_α for all $\alpha < c(X)^+$. Since $c(X)^+ \to (c(X)^+, \omega)$, there exists a chain $\{U_{\alpha_\delta} | \delta < c(X)^+\}$. We can assume $\delta < \delta'$ implies $\alpha_\delta < \alpha_{\delta'}$. Then $\{U_{\alpha_{\delta+1}} - \overline{U_{\alpha_\delta}} | \delta < c(X)^+\}$ is a disjoint collection of open sets of cardinality $c(X)^+$, contradiction. Therefore (*) holds.

Theorem 3. *If* X *is a regular countably compact space with an ortho-base, or a base of subinfinite rank, then* X *is compact.*

Remark. Nyikos [6] has shown that a compact space with an ortho-base is metrizable.

Proof. Let X satisfy the hypothesis of the theorem. If X is also separable, then it is compact by Theorem 2 if it has a base of subinfinite rank, and by Theorem 1 and a theorem of Chaber [2] if it has an ortho-base. If X is not compact, then, X satisfies the hypothesis of Lemma 1. Therefore X contains a perfect pre-image of ω_1, but this is impossible by Lemma 2. Thus X is compact.

A base B for a space X is said to have *point-finite rank* if for each $x \in X$, the set B(x) of all members of B containing x has rank n for some positive integer n. We denote by $r_x B$ the least positive integer n such that B(x) has rank n. Our next theorem answers a question of Nyikos asked in [3].

Theorem 4. *A regular hereditarily Lindelöf space with a base of point-finite rank has a point-countable base.*

Proof. Let X be hereditarily Lindelöf with a base $\mathcal{B}$ of point-finite rank. For each $n \in N$, let $X_n = \{x \in X | r_x \mathcal{B} \leq n\}$. Note that X_n is a closed subset of X.

We follow the proof of [3, Theorem 5.6] to obtain a base $\mathcal{U}_n$ for the points of X_n, such that if $x \in X_n$, then x is in at most countably many elements of $\mathcal{U}_n$. Since the proof is essentially the same, we shall not include it, except to note the following differences:

(i) replace $\mathcal{U}_\alpha$ by $\mathcal{U}'_\alpha = \{U \in \mathcal{U}_\alpha | U \cap X_n \neq \emptyset\}$, and

(ii) replace M_β by $M'_\beta = \{x \in X_n |$ every neighborhood of x contains an element of $\cup_{\alpha<\beta} \mathcal{U}'_\alpha$.

Then the proofs that $X_n = \cup_{\beta<\omega_1} M_\beta$, and that $\cup_{\beta<\omega_1} \mathcal{V}_\beta$ is a base for X_n, point-countable with respect to the points of X_n, are almost exactly the same. We let $\mathcal{U}_n = \cup_{\beta<\omega_1} \mathcal{V}_\beta$.

Since X is hereditarily Lindelöf, for each $U \in \mathcal{U}_n$ there exists a countable set $\mathcal{B}_U \subset \mathcal{B}$ such that $U = \cup\mathcal{B}_U$ and $\overline{B} \subset U$ for each $B \in \mathcal{B}_U$. Let $\mathcal{B}_n = \{B \in \mathcal{B}_U : U \in \mathcal{U}_n, B \cap X_n \neq \emptyset\}$. Then $\mathcal{B}_n$ is a base in X for the points of X_n. Suppose there is a point of X in uncountably many elements of $\mathcal{B}_n$. Then applying $\alpha \rightarrow (\alpha,\omega)^2$, there is an uncountable chain $\mathcal{C} \subset \mathcal{B}_n$. Let $\{C_\alpha\}_{\alpha<\beta}$ be a well-ordered decreasing cofinal subset of minimum cardinality. Then β is not countable, for otherwise there would exist some C_α contained in uncountably many elements of $\mathcal{C}$, contradicting the point-countability of $\mathcal{U}_n$ on X_n. Since X_n is hereditarily Lindelöf, there is a countable subcover of $\{X_n \setminus \overline{C}_\alpha : \alpha < \beta\}$. Thus there is some $\delta < \beta$ with $\cup\{X_n \setminus \overline{C}_\alpha : \alpha < \beta\} = X_n \setminus \overline{C}_\delta$. If $x \in X_n \cap \overline{C}_\delta$, then $x \in \cap_{\alpha<\beta} \overline{C}_\alpha$.

But each $\overline{C}_\alpha$ is contained in some U with $C_\alpha \in \mathcal{B}_U$, so again this contradicts the point-countability of $\mathcal{U}_n$ on X_n. Thus $\mathcal{B}_n$ is point-countable in X, and so $\cup_{n=1}^{\infty} \mathcal{B}_n$ is a point-countable base for X.

In [3], it is proved that a regular separable space with a base of finite rank is metrizable. The following corollary extends this result to the case of a base of point-finite rank.

Corollary. A regular separable space with a base of point-finite rank is metrizable.

Proof. Let X satisfy the hypotheses. Then X is hereditarily Lindelöf by Theorem 2. Thus, by Theorem 4, X has a point-countable base, and by the separability this base is countable.

A space is *proto-metrizable* if it is paracompact and has an ortho-base. Proto-metrizable spaces are monotonically normal [6], and this fact leads to the question asked by Nyikos whether monotonically normal spaces with an ortho-base are paracompact (or, equivalently, proto-metrizable). Here we present an easy solution to this question with the help of a characterization of proto-metrizable spaces due to Gruenhage and Zenor [4]. According to this characterization, a space is proto-metrizable if and only if it has a rank 1 pair-base, i.e., a pair-base $\mathcal{B} = \{B = (B_1, B_2) | B \in \mathcal{B}\}$ such that whenever $B, B' \in \mathcal{B}$ and $B_1 \cap B_1' \neq \emptyset$, then either $B_1 \subset B_2'$ or $B_1' \subset B_2$.

Theorem 5. *A space is proto-metrizable if and only if* *it is monotonically normal and has an ortho-base.*

Proof. We have already noted that a proto-metrizable space has an ortho-base and is monotonically normal. Suppose X is a monotonically normal space with an ortho-base $\mathcal{B}$. By a characterization of monotone normality due to Borges [1], for each open set U and $x \in U$, there exists an open set U_x such that $U_x \cap V_y \neq \emptyset$ implies $x \in V$ or $y \in U$. Let $U_x' = \cap\{B \in \mathcal{B} | x \in B \text{ and } B \not\subset U\}$. Let $U_x'' = U_x \cap U_x'$, and let $\mathcal{P} = \{(U_x'', U) | U \in \mathcal{B}\}$. Suppose $U_x'' \cap V_y'' \neq \emptyset$, where $U, V \in \mathcal{B}$. Without loss of generality, we may assume $x \in V$. If $V \subset U$, then so is V_y''. If $V \not\subset U$, then $U_x' \subset V$, and hence $U_x'' \subset V$. Thus $\mathcal{P}$ is a rank 1 pair-base for X, and so X is proto-metrizable.

References

[1] C. R. Borges, "Four generalizations of stratifiable spaces," *General topology and its relation to modern analysis and algebra III*, (Proceedings Third Prague Topological Sympos. 1971), pp. 73-76. Academia, Prague, 1972.

[2] J. Chaber, *Conditions which imply compactness in countably compact spaces*, to appear.

[3] G. Gruenhage and P. Nyikos, *Spaces with bases of countable rank*, Gen. Top. and Appl., to appear.

[4] ________ and P. Zenor, *Proto-metrizable spaces*, Houston J. Math. 3 (1977).

[5] W. Lindgren and P. Nyikos, *Spaces with bases satisfying certain order and intersection properties*, Pacific J. Math, to appear.

[6] P. Nyikos, "Some surprising base properties in Topology," *Studies in Topology*, Academic Press, New York, 1975.

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