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Let X be a topological space, and let A be a closed subset of X. An open cover $\mathcal{W}$ of X\A is said to be *normal with respect to* A if every homeomorphism h:(X\A) → (X\A) which is limited by $\mathcal{W}$ extends by the identity on A to a homeomorphism on X. (The homeomorphism h is *limited by* $\mathcal{W}$ if and only if for each x ∈ X there is a W ∈ $\mathcal{W}$ such that $\{x,h(x)\} \subset W$.) Let R denote the reals, and let $R^\infty = \text{dir lim } R^n$. In §2 of this paper we show that covers normal with respect to the origin in R^∞ do not exist. In §1 we show how this result arose from the question of stability for R^∞-manifolds. In §3 we indicate that our example works in box topologies as well.

1. Background

Consideration of normal covers for R^∞ arose from the question of stability for R^∞-manifolds. A manifold M modelled on an infinite-dimensional topological vector space (hereafter, TVS) F is said to be *stable* if $M \times F \simeq M$. (Here "≃" denotes "is homeomorphic to.") The problem of whether or not manifolds modelled on a given TVS are stable is one of central importance. For example, the open embedding and classification-by-homotopy-type theorems for manifolds modelled on any of a wide class of normed TVS's was obtained in [2] by combining the stability theorem of R. Schori [10] with work of D. W. Henderson [1]. It is not known whether or not R^∞-manifolds are stable. In [5] it is shown that if M and N are paracompact R^∞-manifolds then (a) $M\times R^\infty$ embeds as an open subset of R^∞, and (b) $M\times R^\infty \simeq N\times R^\infty$ iff M and N have the same homotypy type. Thus, if it were known that R^∞-manifolds were stable, then the open embedding and classification-by-homotopy-type theorems for R^∞-manifolds would follow.

In [6] the author showed that open subsets of R^∞ are stable. That is, if U is any open subset of R^∞, then there is a homeomorphism $h:U\times R^\infty \to U$. Later, as announced in [4], he showed that the homeomorphism h could be taken arbitrarily close to the projection map. One might expect that this result would enable one to obtain the stability theorem for R^∞-manifolds by "meshing" the stability homeomorphisms on neighboring charts in, say, the sprit of A. Jones in [7]. This, however, doesn't seem to work, and the principle reason is the non-existence of normal covers.

2. The Example

The theorem below says precisely that R^∞ does not admit normal covers with respect to $\{0\}$. Here we write 0 for $(0,0,0,\cdots) \in R^\infty$.

Theorem. Let $\mathcal{W}$ *be any open cover of* $R^\infty\backslash\{0\}$. *Then there is a homeomorphism* $h\colon (R^\infty\backslash\{0\}) \to (R^\infty\backslash\{0\})$ *limited by* $\mathcal{W}$ *such that* h *extended to* $h'\colon R^\infty \to R^\infty$ *by* $h'(0) = 0$ *is not continuous.*

Proof. Given real numbers $a < b \leq c < d$ define $\alpha = \alpha(a,b,c,d)\colon R \to R$ and $\beta = \beta(a,c,b,d)\colon R \to R$ by

$$\alpha(x) = \begin{cases} a + (x - a)(\frac{c - a}{b - a}), & x \in [a,b] \\ c + (x - b)(\frac{d - c}{d - b}), & x \in [b,d] \\ x & , x \in ((-\infty,a] \cup [d,\infty)) \end{cases}$$

and

$$\beta(x) = \begin{cases} a + (x - a)(\frac{b - a}{c - a}), & x \in [a,c] \\ b + (x - c)(\frac{d - b}{d - c}), & x \in [c,d] \\ x & , x \in ((-\infty,a] \cup [d,\infty)). \end{cases}$$

It is routine to check that α and β are homeomorphisms with $\beta = \alpha^{-1}$, $\alpha(b) = c$, and $\beta(c) = b$. Further, if $t \in [0,1]$ and $\gamma: R \to R$ is defined by $\gamma(x) = (1 - t)x + t\alpha(x)$, then $\gamma = \alpha(a,b,b + t(c - b),d)$. In particular, γ is a homeomorphism with inverse $\beta(a,b + t(c - b),c,d)$.

For $n \geq 2$ let $a_n = (1/n,0,0,0,\cdots) \in R^\infty$. Choose $W_n \in \mathcal{W}$ such that $a_n \in W_n$. Choose positive numbers $\varepsilon_{n,i}$, $n = 2,3,4,\cdots$, and $i = 1,2,3,\cdots$, such that if

$$C_n = ([1/n - \varepsilon_{n,1}, 1/n + \varepsilon_{n,1}] \times [-2\varepsilon_{n,2}, 2\varepsilon_{n,2}] \times [-2\varepsilon_{n,3}, 2\varepsilon_{n,3}] \times \cdots \times [-2\varepsilon_{n,i}, 2\varepsilon_{n,i}] \times \cdots) \cap R^\infty,$$

then $C_n \subset W_n$. We may do this since sets of the form $(\mathcal{O}_1 \times \mathcal{O}_2 \times \cdots) \cap R^\infty$ where each $\mathcal{O}_i$ is open in R form a basis for R^∞, [3, Prop. II - 1]. We may further pick the $\varepsilon_{n,i}$ such that $\varepsilon_{n,1} < \frac{1}{2}(1/n - 1/(n + 1))$ for all $n \geq 2$ and such that $\varepsilon_{n+1,i} < \frac{1}{2}\varepsilon_{n,i}$ for all $n \geq 2$ whenever $i \geq 2$. Let $\pi_n: R^\infty \to R^\infty$ be defined by

$$\pi_n(x_1,x_2,x_3,\cdots) = (x_1,x_2,\cdots,x_{n-1},x_{n+1},x_{n+2},\cdots), \quad n \geq 2 .$$

Then $\pi_n(C_n) = ([1/n - \varepsilon_{n,1}, 1/n + \varepsilon_{n,1}] \times [-2\varepsilon_{n,2}, 2\varepsilon_{n,2}] \times \cdots \times [2\varepsilon_{n,n-1}, 2\varepsilon_{n,n-1}] \times [2\varepsilon_{n,n+1}, 2\varepsilon_{n,n+1}] \times [2\varepsilon_{n,n+2}, 2\varepsilon_{n,n+2}] \times \cdots) \cap R^\infty$. Since $\pi_n C_n$ contains a neighborhood of a_n and R^∞ is paracompact [5, Prop. III. 1], and hence normal, there is a continuous map $\psi_n: R^\infty \to [0,1]$ such that $\psi_n(a_n) = 1$ and $\psi_n/(R^\infty - \pi_n(C_n)) = 0$. Define $\phi_n: R^\infty \to [0,1]$ by $\phi_n = \psi_n \pi_n$. Then $\phi_n(x)$ is independent of the n^{th} coordinate of x and $\phi_n(a_n) = 1$.

Let $\alpha_n = \alpha(-2\varepsilon_{n,n}, 0, \varepsilon_{n,n}, 2\varepsilon_{n,n})$, $n \geq 2$, be the homeomorphism defined in the first paragraph of this proof. Define $g_n: R^\infty \to R^\infty, n \geq 2$, by

$$g_n(x) = (x_1, x_2, \cdots, x_{n-1}, (1 - \phi_n(x))x_n + \phi_n(x)\alpha_n(x_n), x_{n+1}, x_{n+2}, \cdots) \quad .$$

Note that (a) $g_n(a_n) = (1/n, 0, 0, \cdots, 0, \varepsilon_{n,n}, 0, 0, \cdots)$ and that (b) $g_n|(R^\infty - C_n)$ is the identity since if $x \notin C_n$ then either $\pi_n(x) \notin \pi_n(C_n)$ in which case $\phi_n(x) = 0$, or $x_n \notin [-2\varepsilon_{n,n}, 2\varepsilon_{n,n}]$ in which case $\alpha_n(x_n) = x_n$. Since, as indicated in the beginning of the proof, $(1-\phi_n(x))x_n + \phi_n(x)\alpha_n(x_n) = \alpha(-2\varepsilon_{n,n}, 0, \phi_n(x)\varepsilon_{n,n}, 2\varepsilon_{n,n})(x_n)$ the map g_n is a homeomorphism with inverse $(g_n)^{-1}(x) = (x_1, \cdots, x_{n-1}, \beta(-2\varepsilon_{n,n}, \phi_n(x)\varepsilon_{n,n}, 0, 2\varepsilon_{n,n})(x_n), x_{n+1}, x_{n+2}, \cdots)$.

Choose e_n, $n \geq 2$, such that $(1/(n+1) + \varepsilon_{n+1,1}) < e_n < (1/n - \varepsilon_{n,1})$. Let $\pi: R^\infty \to R$ be the projection $\pi((x_1, x_2, x_3, \cdots)) = x_1$. Define $g: R^\infty \to R^\infty$ by $g/\pi^{-1}([e_2, \infty)) = g_2/\pi^{-1}([e_2, \infty))$, $g/\pi^{-1}([e_n, e_{n-1}]) = g_n/\pi^{-1}([e_n, e_{n-1}])$, $n \geq 3$, and $g/\pi^{-1}((-\infty, 0]) = id$. Note that $g(0) = 0$ and that g is well-defined since if $x_1 = e_n$ then, because of (b) above, $g_n(x) = x = g_{n+1}(x)$. Also, $(g/\pi^{-1}([e_2, \infty)))^{-1} =$

$(g_2/\pi^{-1}([e_2,\infty)))^{-1}$, etc., so that $g/(R^\infty\backslash\pi^{-1}(0)): (R^\infty\backslash\pi^{-1}(0)) \to (R^\infty\backslash\pi^{-1}(0))$ is a homeomorphism. Now, let $r = (0,r_1,r_2,r_3,\cdots) \in (\pi^{-1}(0)\backslash\{0\})$. Choose $k \geq 2$ such that $|r_k| > 0$. Since $\varepsilon_{n+1,k} < \frac{1}{2}\varepsilon_{n,k}$ there is an integer $N > k$ such that $2\varepsilon_{n,k} \leq 2\varepsilon_{N,k} < |r_k|$, all $n \geq N$. Choose positive numbers δ_1 and δ_k such that $\delta_1 < 1/N$ and $\delta_k < (|r_k| - 2\varepsilon_{N,k})$. Let $G = \{y \in R^\infty | \quad |y_1| < \delta_1 \text{ and } |y_k - r_k| < \delta_k\}$. Let $y \in G$. We show $g(y) = y$. This is clear if $y_1 \leq 0$. If $y_1 > 0$ then, since $y_1 < 1/N$, $y \in \pi^{-1}([e_n,e_{n-1}])$ where $n \geq N$. For this n, $2\varepsilon_{n,k} \leq 2\varepsilon_{N,k} \leq |r_k| - \delta_k < |y_k|$. Thus, $y_k \notin [-2\varepsilon_{n,k}, 2\varepsilon_{n,k}]$ so that $y \notin C_n$, and, since $k < n$, $\pi_n(y) \notin \pi_n(C_n))$ so that $\phi_n(y) = \psi_n\pi_n(y) = 0$ and $g(y) = g_n(y) = y$. Thus, G is a neighborhood of r in R^∞ on which g is the identity. Thus g and g^{-1} are continuous at r, and it follows that $h = g/(R^\infty\backslash\{0\}): (R^\infty\backslash\{0\}) \to (R^\infty\backslash\{0\})$ is a homeomorphism.

To see that h is limited by $\mathcal{W}$ let $x \in (R^\infty\backslash\{0\})$. If $x_1 \leq 0$, $h(x) = x$. If $x_1 > 0$, then $h(x) = g_n(x)$ for some n. From (b) above it follows that $g_n/(R^\infty\backslash W_n) = id/(R^\infty\backslash W_n)$. Hence, either $g_n(x) = x$ or $\{g_n(x),x\} \subset W_n$. Thus, h is limited by $\mathcal{W}$. Now extend h to $h': R^\infty \to R^\infty$ by defining $h'(0) = 0$. Let $V = [R \times (-\varepsilon_{2,2},\varepsilon_{2,2}) \times (-\varepsilon_{3,3},\varepsilon_{3,3}) \times \cdots] \cap R^\infty$. Then V is a neighborhood of 0 in R^∞. However, given any neighborhood U of 0 there are positive numbers $\rho_1,\rho_2,\cdots$ such that $U \supset [(-\rho_1,\rho_1) \times (-\rho_2,\rho_2) \times \cdots] \cap R^\infty$, ([3, Prop. II-1]). If $n \geq 2$ is chosen so that $1/n < \rho_1$, then $a_n \in U$ and $h(a_n) = g_n(a_n) = (1/n,0,0,\cdots,0,\varepsilon_{n,n},0,0,\cdots) \notin V$. Thus, h' is not continuous at 0. This completes the proof of the theorem.

3. Remark on Box Topologies

When this paper was presented Scott Williams remarked that the proof would most likely also apply to R^{ω}, by which we denote the countably infinite cartesian product of the real line with the box topology. This is indeed the case. The necessary changes are the obvious ones with one exception. To construct the ϕ_n's above we used the fact that R^{∞} is normal. It is not known whether or not R^{ω} is normal. (In this regard we observe that Mary Ellen Rudin has shown that the continuum hypothesis implies that R^{ω} is paracompact and, hence, normal [9, Theorem 1].) Note, however, that R^{ω} is completely regular [8, p. 49] and that this is sufficient to construct the ϕ_n's.

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