TOPOLOGICAL PROPERTIES OF PRODUCT SPACES AND THE NOTION OF n-CARDINALITY

by

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Recently the author obtained several examples of product spaces with interesting topological properties. In this paper we discuss these examples and the notion of n-cardinality, which plays an important role in their construction.

1. Normality and Paracompactness of Products

Example 1. [P] For every k and m such that $k \leq m \leq \omega$ there exists a separable and first countable space X such that:

(a) X^n is paracompact (Lindelöf, subparacompact) iff $n < k$;

(b) X^n is (collectionwise) normal iff $n < m$.

Corollary 1. [P] *There exists a paracompact space X such that X^2 is normal but X^2 is not paracompact.*

Corollary 1 answers a question raised by E. Michael [M] and attributed by him to M. Maurice. Consistent examples of spaces with properties described in Corollary 1 have been obtained by the author [P_1] in 1973 under the assumption of MA $+\neg$CH and later by K. Alster and P. Zenor [AZ] under the assumption of CH. On the other hand let us recall that the

perfect but X^3 *is not?*

Our next examples give, assuming CH, a positive answer to Heath's question. Let us recall that a Lindelöf space is perfect if and only if it is hereditarily Lindelöf.

Example 3.   (CH) [P_3] For every n < ω there exists a separable and first countable space X such that:

(a) X^n is hereditarily Lindelöf;

(b) X^{n+1} is Lindelöf, but not hereditarily.

Example 4.   (CH) [P_3] For every n < ω there exists a separable and first countable, locally compact and locally countable space X such that:

(a) X^n is perfectly normal;

(b) X^{n+1} is normal, but not hereditarily.

Spaces described in the above examples cannot be both paracompact and locally compact. This follows from the fact that a locally compact and paracompact space whose square is perfect is metrizable (cf. [E]).

4. Subparacompactness of Product Spaces

In 1972, K. Alster and R. Engelking obtained the first example of a paracompact space X whose square is not sub-paracompact [AE]. Their example, however, was neither Lindelöf nor first countable. F. Tall asked [T] whether there exist such Lindelöf spaces and it was also unknown whether products of subparacompact spaces with metric spaces are again subparacompact. These questions are answered by our next example.

Example 5.   [P]   There exists a first countable and

separable Lindelöf space X and a metric separable space M
such that the product space X×M is not countably θ-refinable
(hence, it is not subparacompact).

   Corollary 4.   [P]   *There exists a first countable and*
separable Lindelöf space Z *such that* Z^2 *is not countably*
θ-*refinable.*

   Proof.   Put Z = X ⊕ M.

   The following question remains open.

   Question 2.   *Does there exist a Lindelöf space* X *and a*
complete separable metric space M *such that the product space*
X×M *is not Lindelöf or-equivalently-normal?*

   Such spaces exist if CH is assumed [M].

5. The Notion of n-Cardinality

   In our examples an important role is played by the notion
of n-*cardinality*, which is defined and investigated in $[P_4]$.
The notion of n-cardinality turns out to be useful in con-
structions involving product spaces and seems to be a natural
generalization of the notion of cardinality.

   Definition 1. $[P_4]$ Let X be an arbitrary set and let
A be a subset of X^n, where n is a natural number.   The
n-*cardinality* $|A|_n$ of A (with respect to X^n) is defined by
$|A|_n$ = max $\{|B|: B \subset A$ and $p_i \neq q_i$, for every two distinct
      points $p = (p_1, \cdots, p_n)$ and $q = (q_1, \cdots, q_n)$ from B and
      $i = 1, 2, \cdots, n\}$.
The set A is n-*countable* (n-*uncountable*) if $|A|_n \leq \omega(|A|_n > \omega)$.

   It is shown in $[P_4]$ that $|A|_n$ is well-defined and that:

(1) $|A|_n \leq |A|$;

(2) $|A|_1 = |A|$;

(3) $|A|_2 = \min \{|Y|: Y \subset X \text{ and } A \subset Y \times X \cup X \times Y\}$;

(4) $|A|_n = \min \{|Y|: Y \subset X \text{ and } A \subset \bigcup_{i=1}^{n}(X^{i-1} \times Y \times X^{n-i})\}$ [2].

The following Theorem generalizes a result of van Douwen.

Theorem 1. ($[P_4]$; cf. [vD]) *Let M be a separable complete metric space and B a Borel subset of M^m. The following conditions are equivalent:*

(i) B is n-uncountable;

(ii) $|B|_n = 2^\omega$;

(iii) B contains a homeomorphic image h(C) of the Cantor set C by the diagonal

$$h = \Delta_{i=1}^{n}h_i: C \to M^n$$

of homeomorphic embeddings $h_i: C \to M$.

The following result, which can be derived from Theorem 1, is probably most useful in applications.

Theorem 2. $[P_4]$ *Let M be a separable, complete, and uncountable metric space. There exists a family $\{A_i\}_{i<\omega}$ of disjoint subsets A_i of M such that for every $n < \omega$, for every n-uncountable Borel subset B of M^n and for every $i < \omega$ we have $B \cap A_i^n \neq \emptyset$.*

As an application of Theorem 2, we shall give an independent proof of Corollary 2, i.e.; we shall construct *a space X such that X^n is Lindelöf for all $n < \omega$, but X^ω is not normal.*

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[2] To be precise, equalities (3) and (4) are valid if the n-cardinality of A is infinite.

Construction: Let M be the real line R, and let A_i's be as in Theorem 2. We can clearly assume that $R = \cup_{i<\omega} A_i$. Denote by τ_i the topology on R obtained by making all points from A_i isolated and put $X_i = (R, \tau_i)$ and $X = \oplus_{i<\omega} X_i$.

We shall show that the space X has the required properties.

A. X^n *is Lindelöf, for every* n < ω.

The proof is by induction on n = 0,1,···. Clearly X^0 is Lindelöf. Assume that n > 0, and that X^{n-1} is Lindelöf. In order to show that X^n is Lindelöf it suffices to prove that for every sequence $i_1, ···, i_n$ of indexes the space

$$Y = X_{i_1} \times \cdots \times X_{i_n} = (R^n, \Pi_{j=1}^{n} \tau_{i_j})$$

is Lindelöf. Choose i < ω different from all i_j's, for j = 1,2,···,n. Therefore for every j the topology τ_{i_j} coincides with the usual topology at points of A_i. Let $\mathcal{U}$ be an open covering of Y. There exists a countable refinement $\mathcal{V}$ of $\mathcal{U}$ covering A_i^n and consisting of euclidean-open sets. The set $B = R^n \setminus \cup \mathcal{V}$; is euclidean-closed and disjoint from A_i^n. By Theorem 2 the set B is n-countable and therefore by (4) there exists a countable subset Y of R such that

(5) $B \subset \cup_{j=1}^{n} (R^{j-1} \times Y \times R^{n-j})$.

It follows easily from (5) and the inductive assumption that B is a Lindelöf subspace of Y and hence can be covered by a countable subcollection of $\mathcal{U}$.

B. X^ω *is not normal.*

The space X^ω contains the space $Z = \Pi_{i<\omega} X_i = (R^\omega, \Pi_{i<\omega} \tau_i)$ as a closed subset but the diagonal

$$\Delta = \{(x,x,\cdots) \in R^{\omega}: x \in R\}$$

is a closed subset of Z and since $R = \cup_{i<\omega} A_i$ the subspace Δ

is discrete. This implies that X^{ω} is not Lindelöf. The

non-normality of X^{ω} follows from a theorem of K. Nagami [N]

and P. Zenor [Z] stating that if X^n is Lindelöf, for every

$n < \omega$ and X^{ω} is normal, then X^{ω} is Lindelöf. This completes

the proof.

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