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by

T. B. Muenzenberger and R. E. Smithson

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Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
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SEMILATTICE STRUCTURES ON DENDRITIC SPACES

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1. Introduction

The algebraic structure available in dendritic spaces is refined and studied in a spirit analogous to that found in [5], [6], and [7]. A major contribution is to fill a gap in the proof of a key lemma in [14] by proving that the cutpoint order gives a dendritic space the structure of a meet semilattice. The actual mechanics involve the theory of Galois correspondences and gap points in partially ordered sets. The assumption of rim finiteness on a dendritic space yields an even finer algebraic structure on the space.

For example, it is shown that a rim finite dendritic space has the structure of a monotone topological semilattice and that the topology on such a space is uniquely determined by the cutpoint order. The paper also includes another construction of the unique dendritic compactification of a rim finite dendritic space. Other contributions to the theory of dendritic spaces can be found in the work of Allen [1], Bennett [2], Gurin [4], Pearson [10], Proizvolov [11] and [12], and Ward [13] and [14].

2. Galois Correspondences and Gap Points in Posets

Let $(X,\leq)$ be a poset. For $A \subset X$ define $L(A) = \{x \in X | a \in A \Rightarrow x \leq a\}$ and $M(A) = \{x \in X | a \in A \Rightarrow a \leq x\}$. For $x \in X$ define $L(x) = L(\{x\})$ and $M(x) = M(\{x\})$.

Lemma 2.1. *Let* $(X,\leq)$ *be a poset, let* $x \in X$*, and let* $A,B \subset X$. *Then*

a. $A \subset LM(A)$ *and* $A \subset ML(A)$.

b. *If* $A \subset B$*, then* $L(B) \subset L(A)$ *and* $M(B) \subset M(A)$.

c. $A \subset L(B)$ *if and only if* $B \subset M(A)$.

d. *If* $A \subset B$*, then* $LM(A) \subset LM(B)$ *and* $ML(A) \subset ML(B)$.

e. $LML(A) = L(A)$ *and* $MLM(A) = M(A)$.

f. $LM(x) = L(x)$ *and* $ML(x) = M(x)$.

g. *If* $x \in L(A)$*, then* $L(x) \subset L(A)$. *If* $x \in M(A)$*, then* $M(x) \subset M(A)$.

As observed in [3] and [9], statements (a) and (b) taken together say that L and M are *Galois correspondences* for the subsets of X. Statement (c) is an equivalent formulation due to J. Schmidt [3, page 124]. Statements (d) and (e) are proved in [3] and [9]. Statement (f) is alluded to on page 126 of [3], and statement (g) follows easily from transitivity.

If $(X,\leq)$ is a poset and $X_o \subset X$, then $x \in X_o$ is a *gap point* of X_o if and only if

$$(L(x) - \{x\}) \cap ML(X_o) = \emptyset.$$

Lemma 2.2. *Let* $(X,\leq)$ *be a poset and let* $X_o \subset X$. *Then*

a. *If* $x \in X_o \cap L(X_o)$*, then* x *is a gap point of* X_o.

b. *No two gap points of* X_o *are comparable.*

c. *If* x *is a gap point of* X_o *and* $x \in Y_o \subset ML(X_o)$*, then* x *is a gap point of* Y_o.

Proof. (a) If $x \in X_o \cap L(X_o)$, then $ML(X_o) = ML(x) = M(x)$ and so x is a gap point of X_o. (b) If x and x' are gap points of X_o with $x < x'$, then $x \in (L(x') - \{x'\}) \cap ML(X_o) = \emptyset$.

(c) If $Y_o \subset ML(X_o)$, then $ML(Y_o) \subset ML(X_o)$ from which the result follows readily.

Lemma 2.3. *Let* $(X,\leq)$ *be a poset and let* $X_o \subset X$ *be a nondegenerate set of noncomparable elements. Then*

a. $X_o \cap L(X_o) = \emptyset$.

b. If x *is a gap point of* X_o *and* $L(x)$ *is a toset, then* $L(X_o) = L(x) - \{x\}$.

Proof. (a) An element of $X_o \cap L(X_o)$ would be comparable to any element of X_o. (b) $L(X_o) \subset L(x) - \{x\}$ since $x \in X_o - L(X_o)$. Suppose that $y < x$. Since x is a gap point of X_o, there exists $a \in L(X_o)$ such that $a \not\leq y$. So $y < a$ and thus $y \in L(X_o)$.

3. Mods

Let $(X,\leq)$ be a poset and for $x,y \in X$ define $x \wedge y = \inf \{x,y\}$ when the infimum involved exists.

Definition 3.1. A poset $(X,\leq)$ is a *mod* if and only if the following conditions hold:

a. For all $x,y \in X$, $x \wedge y$ exists.

b. For all $x \in X$, $L(x)$ is a toset.

c. Each nonempty subset of X which is bounded above (below) has a supremum (an infimum) in X.

d. If $x,y \in X$ and $x < y$, then there exists $z \in X$ such that $x < z < y$.

So a mod is a conditionally complete and order dense meet semilattice in which the lower sets are tosets.

There is an alternative approach. Let X be a set, let $\mathcal{P}$ be a collection of subsets of X, and consider the following five axioms on $(X,\mathcal{P})$.

Axiom 1. *If* $x,y \in X$, *then there exists* $P \in \mathcal{P}$ *such that* $x,y \in P$.

Axiom 2. *If* $\emptyset \neq \mathcal{P}_o \subset P$ *and if* $\cap \mathcal{P}_o \neq \emptyset$, *then* $\cap \mathcal{P}_o \in \mathcal{P}$.

If $x,y \in X$, then the *chain* with endpoints x and y is defined by $[x,y] = \cap\{P \in \mathcal{P} | x,y \in P\}$. Set $[x,y) = [x,y] - \{y\} = (y,x]$ and $(x,y) = [x,y] - \{x,y\}$. A set $A \subset X$ is *chainable* if and only if $x,y \in A$ imply $[x,y] \subset A$.

Axiom 3. *If* $P \in \mathcal{P}$, *then there exist unique* $x,y \in X$ *such that* $P = [x,y]$.

Axiom 4. *The union of two chains that meet is chainable.*

Axiom 5. *If* $x,y \in X$ *and* $x \neq y$, *then* $(x,y) \neq \emptyset$.

If e is an element of X, then the *chain order* $\leq$ with *basepoint* e is defined by the rule: If $x,y \in X$, then $x \leq y$ if and only if $x \in [e,y]$.

Theorem 3.2. *If* $(X,\mathcal{P})$ *satisfies Axioms* 1-5, *if* $e \in X$, *and if* $\leq$ *is the chain order with basepoint* e, *then* $(X,\leq)$ *is a mod with least element* e *and the chains obey the following equality:*

$$[x,y] = \begin{cases} L(y) \cap M(x) & \text{if } x \leq y. \\ [x \wedge y, x] \cup [x \wedge y, y] & \text{if } x \text{ and } y \text{ are not comparable.} \end{cases}$$

A proof of Theorem 3.2 can be found in [5].

Theorem 3.3. *If* $(X,\leq)$ *is a mod and the chains in* $\mathcal{P}$ *are defined as in the previous theorem, then* $(X,\mathcal{P})$ *satisfies*

Axioms 1-5. *Moreover, if* f $\in$ X, *then the chain order with basepoint* f *is exactly*

$(\leq - \{(x,y) \in X \times X | x < f \wedge y\}) \cup \{(x,y) \in X \times X | f \wedge y \leq x \leq f\}$.

A proof of the first part of Theorem 3.3 may be found in [5]. The second part of Theorem 3.3 is not necessary for the primary purposes of this paper, and so the proof is omitted. Theorems 3.2 and 3.3 show that the concept of a mod is coextensive with that of a mod with least element.

Many results proved in [6] did not require the existence of maximal elements, and so they are valid in the more general context of a mod. Here is an example. An *arc* is a Hausdorff continuum with exactly two noncutpoints. A Hausdorff space X is *acyclic* if and only if any two distinct points $x,y \in X$ are the endpoints of a unique arc A[x,y] in X. An acyclic space admits a natural mod structure which will now be described. Define $A[x,x] = \{x\}$ where $x \in X$. If X is an acyclic space and $e \in X$, then the *arc order* $\leq$ with *basepoint* e is defined by the rule: If $x,y \in X$, then $x \leq y$ if and only if $x \in A[e,y]$. It was shown in [8] that $(X,\leq)$ is a mod. In fact, there is a topological characterization of mods with certain natural order compatible topologies in terms of acyclic spaces which is analogous to Theorems 7.1 and 7.2 in [6].

If $(X,\leq)$ is a mod with least element and each nonempty toset in X has a supremum in X, then $(X,\leq)$ is called a *semitree*. Semitrees were studied extensively in [5]-[7].

4. Dendritic Spaces

Let X be a connected Hausdorff space. If e is an element of X, then the *cutpoint order* $\leq$ with *basepoint* e is defined by the rule: If $x,y \in X$, then $x \leq y$ if and only if $x = e$, $x = y$, or x separates e and y. Let Max(X) denote the $\leq$ maximal elements of X and let N(X) denote the noncutpoints of X.

Lemma 4.1. *Let* X *be a connected Hausdorff space, let* $e \in X$, *and let* $\leq$ *be the cutpoint order with basepoint* e. *Then*

a. $\leq$ *is a partial order with least element* e.

b. *For all* $x \in X$, L(x) *is a toset.*

c. *For all* $x \in X$, $M(x) - \{x\}$ *is an open set.*

d. *If* $x,y,z \in X$ *and* z *separates* x *and* y, *then* $z < x$ *or* $z < y$.

e. *If* X *is nondegenerate, then* $\text{Max}(X) = N(X) - \{e\}$.

Proof. Statements (a)-(c) are proved in [14]. (d) Suppose that $X - \{z\} = A \cup B$ where $x \in A$, $y \in B$, and A and B are separated sets. If $z = e$, then $z < x,y$. If $z \neq e$, then $z < x$ or $z < y$ depending on whether $e \in B$ or $e \in A$. (e) See the proof of Theorem 13 in [14].

A *dendritic space* is a connected Hausdorff space in which each pair of distinct points can be separated by some third point. Through Proposition 4.7 in this section, X will denote a dendritic space, e will denote an element of X, and $\leq$ will denote the cutpoint order with basepoint e. The following Lemma is also proved in [14].

Lemma 4.2. *(a) The partial order* $\leq$ *is dense. (b) For all* $x \in X$, L(x) *and* M(x) *are closed sets. (c) The partial* *order* $\leq$ *is a closed subset of* $X \times X$.

The remainder of what is needed to show that $(X,\leq)$ is a mod would seem to be provided by Lemma 8.2 of [14]. Unfortunately, there is a gap in the proof of Lemma 8.2 given therein because, in the notation of the proof of that lemma, x_1 and y need not be comparable. In effect, what is needed to complete the proof of the Lemma is the existence of $x_1 \wedge y$, and the main part of this section will be devoted to proving the existence of the infimum of a doubleton in X.

Lemma 4.3. *If* $X_o \subset X$ *is a set of noncomparable elements, then there exists at most one gap point of* X_o.

Proof. Suppose that x_1 and x_2 are distinct gap points of X_o. By Lemma 2.3, $[e,x_1) = L(X_o) = [e,x_2)$. Let $z \in X$ separate x_1 and x_2. By Lemma 4.1, it may be assumed that $z < x_1$. Then $z \notin ML(X_o)$ since x_1 is a gap point of X_o. So there exists $a \in L(X_o)$ such that $a \not\leq z$. But $a,z < x_1$ whence $z < a$ since $L(x_1)$ is a toset. Now there are two separations of interest here. First, $X - \{z\} = C \cup D$ where $x_1 \in C$, $x_2 \in D$, and C and D are separated sets. Second.

$$X - \{a\} = (X - M(a)) \cup (M(a) - a)$$

where $e,z \in X - M(a)$, x_1 and x_2 are in $M(a) - \{a\}$, and $X - M(a)$ and $M(a) - \{a\}$ are disjoint open sets. The question is where does a lie in the first separation? If $a \in C$, then $D \cup \{z\}$ is connected and meets both pieces of the second separation. Similarly, $a \in D$ leads to a contradiction. So X_o has at most one gap point.

Corollary 4.4. *If* $X_o \subset X$, *then there exists at most one* *gap point of* X_o.

Proof. Suppose that x_1 and x_2 are distinct gap points of X_o. Then x_1 and x_2 are gap points of $Y_o = \{x_1, x_2\} \subset X_o$ by Lemma 2.2(c). By Lemma 2.2(b), x_1 and x_2 are noncomparable. Then Lemma 4.3, when applied to Y_o, provides a contradiction.

Theorem 4.5. *If* $x_1, x_2 \in X$, *then* $x_1 \wedge x_2$ *exists.*

Proof. Let $x_1, x_2 \in X$ and define $B = ML\{x_1, x_2\}$. Observe that $ML(B) = B$ by Lemma 2.1(e), that

$$B = \cap\{M(a) \mid a \in L\{x_1, x_2\}\}$$

is a closed set by Lemma 4.2(b), and that B has at most one gap point by Corollary 4.4. Suppose first that B has no gap points. Then

(1) $B = \cup\{M(b) - \{b\} \mid b \in B\}$.

For suppose that $x \in B$. Then $[e,x) \cap B \neq \emptyset$ since x is not a gap point of B, and if $b \in [e,x) \cap B$, then $x \in M(b) - \{b\}$ where $b \in B$. On the other hand, if $x \in M(b) - \{b\}$ where $b \in B$, then $x \in M(b) \subset B$ by Lemma 2.1(g). By (1) and Lemma 4.1(c), B is an open set. Thus, B is a clopen set, and therefore $B = X$. So $e \in B$, but e is a gap point of X by Lemma 2.2(a). Thus, B must have exactly one gap point. Suppose that b_o is the only gap point of B. Then

(2) $B - \{b_o\} = \cup\{M(b) - \{b\} \mid b \in B\}$.

The proof of (2) is identical with the proof of (1) save for the observation that $b_o \notin M(b) - \{b\}$ when $b \in B$ because $[e,b_o) \cap B = \emptyset$. So $B - \{b_o\}$ is an open set. If $e \in B$, then $e = x_1 \wedge x_2$. Suppose that $e \notin B$. If $x_1 = x_2$, then $x_1 = x_1 \wedge x_2 = x_2$. Suppose that $x_1 \neq x_2$. Then

(3) $X - \{b_o\} = (X - B) \cup (B - \{b_o\})$

is a separation. If $b_o = x_1$, then (3) implies that $x_1 < x_2$ and $x_1 = x_1 \wedge x_2$. Similarly, $b_o = x_2$ implies that $x_2 = x_1 \wedge x_2$. Finally, if $x_1 \neq b_o \neq x_2$, then (3) implies that $b_o < x_1, x_2$ and so $b_o = x_1 \wedge x_2$.

Corollary 4.6. *The pair* $(X, \leq)$ *is a mod.*

Proof. There are two avenues of attack here. First, it can be proven directly from Theorem 4.5 that each nonempty subset of X has an infimum in X, and from that it can be easily proven that each nonempty subset of X which is bounded above has a supremum in X. Second, the gap in the proof of Lemma 8.2 of [14] has now been filled, and so that Lemma may also be applied to yield the desired conclusion.

If X is a dendritic space and $X_0 \subset X$, then inf X_0 is a gap point of $ML(X_0)$ and vice versa. In fact, the concepts of gap point and infimum can be made to coincide in a dendritic space by changing the definition of gap point to only require that a gap point x of X_0 be a member of $ML(X_0)$ instead of requiring that x lie in X_0 itself.

Theorem 4.5 also provides simple proofs of several results in [14]. Here is an example.

Proposition 4.7. *For all* $x \in X$, $L(x)$ *is closed.*

Proof. Let $y \notin L(x)$ and choose t so that $x \wedge y < t < y$. Then $y \in M(t) - \{t\} \subset X - L(x)$.

In fact, for any $x \in X$

$$X - L(x) = \cup\{M(t) - \{t\} \mid t \in X \text{ and } t \not< x\}.$$

For another application, note that the existence of mod structures on acyclic and dendritic spaces means that the fixed point theorems proven in [5], [6], [8], and [14] are valid for such spaces.

The final theorem in this section can be proved by using Theorem 3.3, Lemmas 4.1 and 4.2, and Proposition 2.3 in [14].

Theorem 4.8. *Let* X *be an acyclic or a dendritic space, let* $e \in X$, *and let* $\leq$ *be the arc order or the cutpoint order with basepoint* e, *respectively. Then the family* $\mathcal{P}$ *of chains described in Theorem* 3.3 *is independent of* e. *In fact, if* $f \in X$, *then the chain order with basepoint* f *determined by* $\mathcal{P}$ *equals the arc order or the cutpoint order with basepoint* f, *respectively.*

5. Rim Finite Dendritic Spaces

If A is a subset of a space X, then $\bar{A}$ will denote the closure of A in X. A space X is *rim finite* if and only if each element of X admits arbitrarily small neighborhoods whose boundaries are finite. Through Theorem 5.11, let X be a rim finite dendritic space and let X* denote the unique dendritic compactification of X first given by Proizvolov [11] and later refined by Allen [1], Pearson [10], and Ward [14]. By the construction of X* given in [14],

$$X^* = X \cup N_o \text{ where } X \cap N_o = \emptyset \text{ and } N_o \subset N(X^*).$$

This is because the cutpoints of X* are all members of X as observed in the proof of Theorem 23 in [14]. Let $e \in X$, let $\leq$ be the cutpoint order on X with least element e, and let $\leq^*$ be the cutpoint order on X* with least element e. Let $\wedge$ and $\wedge^*$ be the corresponding infimum operations. By Corrollary 4.6, $(X,\leq)$ and $(X^*,\leq^*)$ are mods. Since X* is a *tree* (a compact dendritic space), $(X^*, \leq^*)$ is actually a semitree, and $\mathcal{J}^*$, the topology of X^*, is *strongly order compatible* as defined in [6]. The later fact is well known, and indeed it follows from Lemmas 4.1 and 4.2, Theorem 4.8, and certain results in [6]. Thus, if $x,y \in X^*$ and $[x,y]^*$ denotes the chain in X^* with endpoints x and y, then $[x,y]^*$ is the unique arc in X^* with endpoints x and y. Furthermore, the family consisting of all sets of the form

$$M^*(x) - \{x\},\ X^* - M^*(x),\ X^* - L^*(x) \text{ where } x \in X^*,$$

$$L^*(x) = \{y \in X^* | y \leq^* x\}, \text{ and } M^*(x) = \{y \in X^* | x \leq^* y\}$$

is a subbasis for $\mathcal{J}^*$ by Theorem 4.27 in [6]. The purpose of the present section is to extend these and other results to X in so far as possible.

Lemma 5.1. *If* $m \in \text{Max}(X^*)$, *then the family*

$$\mathcal{J} = \{M^*(t) - \{t\} | t \in X^* \text{ and } t <^* m\}$$

is a basis for the neighborhoods of m *in* X^*.

Proof. Let $x \in X^*$ and $m \in X^* - M^*(x)$. If $x \wedge^* m <^* t <^* m$, then $m \in M^*(t) - \{t\} \subset X^* - M^*(x)$. The lower sets have already been handled in Proposition 4.7. Furthermore, $\mathcal{J}$ is closed under finite intersections, and so $\mathcal{J}$ is an open basis at m in X^*.

Corollary 5.2. *If* $x \in X^*$, *then a separation* $X^* - \{x\} = C \cup D$ *of* $X^* - \{x\}$ *restricts to a separation* $X - \{x\} = (C \cap X) \cup (D \cap X)$ *of* $X - \{x\}$.

Proof. If $C \cap X = \emptyset$, then $C \subset N_0 \subset \text{Max}(X^*)$. But C is open, contradicting Lemma 5.1 or else the local connectivity of X^*.

Lemma 5.3. *The inclusion* $\leq^* \cap (X \times X) \subset \leq$ *holds.*

Proof. Apply Corollary 5.2.

Actually, Corollary 5.2 is not necessary to prove Lemma 5.3, but Corollary 5.2 will be needed later in the proof of Corollary 5.8.

Lemma 5.4. *If* $C \subset X^*$ *is chainable and* $M \subset \mathrm{Max}(X^*)$, *then* $C - M$ *is chainable and therefore connected.*

Proof. If $x,y \in X^* - M$, then $[x,y]^* \subset L^*(x) \cup L^*(y) \subset X^* - M$. Hence, $C - M$ is chainable and therefore connected by Theorem 4.21 in [6].

Corollary 5.5. $\mathrm{Max}(X^*) = \mathrm{Max}(X) \cup N_o$.

Proof. Apply Lemmas 4.1(e), 4.2(a), 5.3, and 5.4.

All closures in the next Lemma and proof are to be taken in X^*.

Lemma 5.6. *If* $x \in X$, *then a separation* $X - \{x\} = C \cup D$ *of* $X - \{x\}$ *extends to a separation* $X^* - \{x\} = (\overline{C} - \{x\}) \cup (\overline{D} - \{x\})$ *of* $X^* - \{x\}$.

Proof. Notice that $\overline{C} \cap \overline{D} = \{x\}$ by Lemma 1.2 in [1], and so $\overline{C} - \{x\}$ and $\overline{D} - \{x\}$ are separated sets in $X^* - \{x\}$. Now $\overline{X - \{x\}} = X^*$, and so $X^* - \{x\} = (\overline{C} - \{x\}) \cup (\overline{D} - \{x\})$ is a separation.

Corollary 5.7. *The partial order* $\leq^*$ *restricted to* $X \times X$ *equals* $\leq$.

Of course Corollary 5.7 follows immediately from Lemmas 5.3 and 5.6, but alternate proofs are possible by verifying the conditions in Theorem 5 or Corollary 11.1 or 12.1 in [14].

Corollary 5.8. $N(X^*) = N(X) \cup N_o$.

Proof. Lemmas 4.1(e) and 5.6 and Corollary 5.5 are enough to show that $N(X^*) \subset N(X) \cup N_o$, and Corollary 5.2 shows that $N(X) \subset N(X^*)$.

Corollary 5.9. *The equation* $\wedge^*|_{X\times X} = \wedge$ *holds.*

Proof. This is immediate from Corollary 5.7.

Theorem 5.10. *The pair* $(X,\wedge)$ *is a monotone topological semilattice.*

Proof. The function $\wedge^*$ is continuous since X^* is a tree. So $\wedge$ is continuous by Corollary 5.9, and $\wedge$ is shown to be monotone in the proof of Theorem 11 in [7].

Examples 3 and 4 in [7] show that an arbitrary dendritic space need not have a continuous infimum operation. However, Theorem 11 in [7] does yield a result about the continuity of the infimum operation on certain dendritic spaces that is slightly better than the one in Theorem 5.10. To see that it is a better result necessitates a fuller illumination of the structure of a rim finite dendritic space.

It seems to be well known that a rim finite dendritic space X is acyclic; see, for example, [2], [4], [10], [11], and [14]. In fact, if $x,y \in X$, then $[x,y] = [x,y]^*$ is the unique arc joining x and y by Lemma 5.7, Corollary 5.9, and the corresponding structure on X^*. Thus, the cutpoint order on X with basepoint e coincides with the arc order on X with basepoint e. Compare Theorem 4.8.

Theorem 5.11. *The family consisting of all sets of the form*

$$M(x) - \{x\},\ X - M(x),\ X - L(x)$$

where x ∈ X *is a subbasis for the topology* $\mathcal{J}$ *of* X.

Proof. The subbasis for $\mathcal{J}^*$ described at the beginning of this section restricts to a subbasis for $\mathcal{J}$. However, for $x \in X^*$

$$(M^*(x) - \{x\}) \cap X = \begin{cases} M(x) - \{x\} & \text{if } x \in X. \\ \emptyset & \text{if } x \notin X. \end{cases}$$

$$(X^* - M^*(x)) \cap X = \begin{cases} X - M(x) & \text{if } x \in X. \\ X - \{x\} = X & \text{if } x \notin X. \end{cases}$$

$$(X^* - L^*(x)) \cap X = \begin{cases} X - L(x) & \text{if } x \in X. \\ \cup\{M(t) - \{t\} \mid t \in X,\ t \not\leq^* x\} & \text{if } x \notin X. \end{cases}$$

The last equality holds because of the equality following Proposition 4.7. So the family described in the statement of the theorem is also a subbasis for $\mathcal{J}$.

Let $(X,\mathcal{J})$ be a connected space with topology $\mathcal{J}$. Following Ward [14], let σ denote the topology generated by the family of all components of sets of the form X - {x} where x ∈ X. Suppose now that $(x,\leq)$ is a mod and let $\mathcal{J}_o$ denote the topology generated by the family of all sets of the form

$$M(x) - \{x\},\ X - M(x), \text{ and } X - L(x) \text{ where } x \in X.$$

The topology $\mathcal{J}_o$ is called the *tree topology* determined by $\leq$.

Corollary 5.12. *If* $(X,\mathcal{J})$ *is a dendritic space, if* e ∈ X, *and if* $\leq$ *is the cutpoint order with basepoint* e, *then the following statements are equivalent.*

a. $(X,\mathcal{J})$ *is rim finite.*

b. $\mathcal{J} = \sigma$.

c. $\mathcal{J} = \mathcal{J}_o$, *the tree topology determined by* $\leq$.

Proof. (a) and (b) were shown to be equivalent in

Theorem 21 in [14]. (a) ⇒ (c) is Theorem 5.11. (c) ⇒ (b) Observe that the components of X - {x} where x ∈ X consist of X - M(x) and all B - {x} where B is a *branch* at x as defined in [6]. Apply Lemma 4.19 in [6] and the equality following Proposition 4.7.

Several other conditions which are equivalent to rim finiteness in a dendritic space are given in [14].

Theorem 5.13. *Let* $(X,\leq)$ *be a mod with least element* e *and let* $\mathcal{T}_o$ *be the tree topology determined by* $\leq$. *Then* $\mathcal{T}_o$ *is a strongly order compatible topology and* $(X,\mathcal{T}_o)$ *is a rim finite dendritic space. Moreover,* $\mathcal{T}_o$ *is the only rim finite dendritic order compatable topology on* X.

Proof. Let $\mathcal{P}$ denote the family of chains in X, let $\mathcal{T}(\mathcal{P})$ denote the strong topology on X induced by $\mathcal{P}$, and let $\mathcal{I}$ denote the interval topology on X (see [6]). Then $\mathcal{I} \subset \mathcal{T}_o \subset \mathcal{T}(\mathcal{P})$ by definition of $\mathcal{T}_o$ and Lemma 4.3 in [6]. Hence, $\mathcal{T}_o$ is a strongly order compatible topology. Now $(X,\mathcal{T}_o)$ is dendritic by Lemmas 4.19 and 4.25 and Theorem 4.21 in [6], and $(X,\mathcal{T}_o)$ is easily seem to be rim finite. Let $\mathcal{T}$ be an order compatible topology on X for which $(X,\mathcal{T})$ is a rim finite dendritic space. Then for x,y ∈ X, the chain [x,y] is an arc in $(X,\mathcal{T})$, whereas the space $(X,\mathcal{T})$ is acyclic as observed earlier. Accordingly, the cutpoint order on $(X,\mathcal{T})$ with basepoint e, the arc order on $(X,\mathcal{T})$ with basepoint e, and the chain order on $(X,\mathcal{P})$ with basepoint e (namely, $\leq$) all coincide. Corollary 5.12 then implies that $\mathcal{T} = \mathcal{T}_o$.

So there do not exist two nonhomeomorphic rim finite dendritic spaces having order isomorphic mod structures compatible with their respective topologies. When applied to an acyclic space with an arc order, Theorem 5.13 shows that the rim finite dendritic spaces play the same role amongst the acyclic spaces that the trees play amongst the nested spaces [6].

There is another construction of X* which is identical as a set to that provided by Ward in [14], but which differs from his construction in that the topology is derived from the order. Let $(X,\mathcal{J})$ be a rim finite dendritic space, let $e \in X$, and let $\leq$ be the cutpoint order with basepoint e. Then $(X,\leq)$ is a mod and $\mathcal{J} = \mathcal{J}_o$ is the tree topology determined by $\leq$. For each maximal $\leq$ toset T in X with no largest element, adjoin to X an element $x_T \notin X$. Let $N_o = \{x_T | T$ is a maximal $\leq$ toset in X with no largest element$\}$ and let $X^* = X \cup N_o$. Extend $\leq$ to X* as follows.

a. If $x,y \in X$, then $x \leq^* y$ if and only if $x \leq y$.

b. If $x,y \in N_o$, then $x \leq^* y$ if and only if $x = y$.

c. If $x \in X$ and $y \in N_o$, then $x \leq^* y$ if and only if $x \in T$, a maximal $\leq$ toset in X with no largest element, and $y = x_T$.

Then $(X^*,\leq^*)$ is a semitree and $\leq^* \cap (X \times X) = \leq$. If $\mathcal{J}_o^*$ is the tree topology determined by $\leq^*$, then $(X^*,\mathcal{J}_o^*)$ is compact Hausdorff and dendritic by Theorem 4.27 in [6]. It can be shown that $\mathcal{J}_o^* \cap X$ is a rim finite dendritic order compatible topology on X, and hence, $\mathcal{J}_o^* \cap X = \mathcal{J}_o$ by Theorem 5.13. But the later equality is more readily derivable from the equalities listed in the proof of Theorem 5.11, and those equalities are easily shown to be valid in the present context.

Observe that the construction of X* given herein really does depend on the construction provided by Ward in [14] since Theorem 5.11 required the existence of X* in its proof and was used in observing that $\mathcal{J} = \mathcal{J}_o$.

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Kansas State University

Manhattan, Kansas 66506

and

University of Wyoming

Laramie, Wyoming 82071