SPACES WHICH ARE SCATTERED WITH RESPECT TO COLLECTIONS OF SETS

by

H. H. Wicke and J.M. Worrell, Jr.

SPACES WHICH ARE SCATTERED WITH
RESPECT TO COLLECTIONS OF SETS

H. H. Wicke and J.M. Worrell, Jr.

We study in this paper some of the topological structure associated with scattered spaces, σ-scattered spaces, or, more generally, spaces which are collectionwise or σ-collectionwise scattered with respect to collections of sets which are either complete or satisfy some sort of generalized uniform first countability condition. A number of important counterexamples are spaces of this kind. One of the earliest results known to us of the type considered is that of Kuratowski [K]: Every separable metric scattered space is an absolute G_δ. Telgársky [T] proved that every T_2 paracompact first countable scattered space is an absolute metric G_δ. The authors [WW$_2$] showed that every T_1 first countable scattered space has a λ-base. (Both the results of Kuratowski and Telgársky follow from this theorem which is a point of departure for the present paper.) Subsequently they showed (announced in [WW$_7$]) that σ-(closed and scattered) T_1 first countable spaces have a base of countable order and σ-scattered first countable spaces have primitive bases. Furthermore, by replacing *first countable* by *point-countable type* or by q-*space*, analogues of the above results are obtained involving the β_b, β_c, λ_b, λ_c-spaces [W$_1$] and primitively quasi-complete spaces. These results will be obtained here as special cases of theorems involving generalizations of

scattered and σ-*scattered*. The main concept is that of a
space being scattered with respect to a collection $\mathcal{M}$ of its
subsets (Definition 2.1). In this generalization, the points
of scattered spaces are replaced by certain sets having suit-
able properties: for example, each M ∈ $\mathcal{M}$ is compact and of
countable character or each M ∈ $\mathcal{M}$ has a base of countable
order in the global topology. An analogue of the first
theorem stated above is: Suppose X is $\mathcal{M}$-collectionwise
scattered with respect to a collection $\mathcal{M}$ of sets having a
base of countable order (λ-base) in the global topology.
Then X has a base of countable order (λ-base). The various
local-implies-global theorems of base of countable order
theory [WW$_8$] are special cases of the results obtained here.

    Passing from *scattered* to *collectionwise scattered*
greatly widens the scope of the results previously obtained.
Simple examples show this: No uncountable separable metric
connected space can be σ-scattered; however such a space
can be σ-$\mathcal{M}$-collectionwise scattered, where $\mathcal{M}$ is a disjoint
collection of arcs.

    We present first the main definition and some consequences.
In order to derive a number of results efficiently, we then
prove some theorems concerning spaces which are scattered with
respect to collections of sets each of which has an open primi-
tive sequence in the global topology. The results indicate
that the primitive sequence theory is a natural setting for
the discussion of collectionwise scattered spaces. We then
apply these theorems to get results concerning some of the
concepts of base of countable order theory including those
mentioned above. Examples show that these results cannot be

improved, say, by requiring collectionwise scattered with
respect to sets which are metrizable in the global topology.

   Terminology.   A space is *scattered* if and only if it
contains no nonempty dense-in-itself subset [K]; it is
called σ-*scattered* if it is the union of a countable collec-
tion of scattered subsets; if each of these scattered sub-
sets is closed, the space is called σ-*(closed and scattered)*.

   As an heuristic guide, we emphasize that for first
countable T_1 spaces, scattered implies a strongly complete
base of countable order [WW$_2$], σ-(closed and scattered) im-
plies base of countable order [WW$_8$], and σ-scattered implies
primitive base [WW$_8$].   (Proofs of these statements are given
in Section 7 below.)   Analogues of these results hold for
collectionwise scattered spaces and this is the underlying
organizing principle of the presentation; see especially
Sections 4 and 5.   The replacement of points by spaces having
certain structures in the global topology leads to Theorems
6.1, 6.3, and 6.6 which are generalizations of the theorems
just quoted (in the order as given).   We note that a number
of the results of Section 7 were presented at the Prague
Conference [WW$_8$]; a summary is to appear in the Proceedings
of the Conference.

2. Collectionwise Scattered Sets

   The definition of collectionwise scattered sets is given
here and various properties are established.

   Definition 2.1.   Let X be a space, A $\subseteq$ X and $\mathcal{M}$ a col-
lection of subsets of X.   Then A is called $\mathcal{M}$-*collectionwise*

scattered in X if and only if there exists $N \subseteq M$ such that

A = $\cup N$ and for all $K \subseteq N$, if $K \neq \emptyset$, there exist M $\in K$ and an

open set U_M such that M $\subseteq U_M$ and for all K $\in K$, K $\cap U_M \subseteq$ M.

   Theorem 2.2. A set A $\subseteq$ X *is scattered in the usual*

sense if and only if it is M-collectionwise scattered, where

M = {{x}: x $\in$ A}.

   Theorem 2.3. Suppose A $\subseteq$ X *has property* P *locally*

(i.e. for all x $\in$ A *there is an open* U *in* X *such that* x $\in$ U

and U $\cap$ A *has* P). *Then if M is the set of all subsets of*

X *having* P, *the set* A *is M-collectionwise scattered in* X.

   Proof. Let N = {U $\cap$ A: U is open in X and U $\cap$ A has P}.

Then A = $\cup N$. If U $\cap$ A, V $\cap$ A $\in N$ then U $\cap$ A $\subseteq$ U and

(V$\cap$A) $\cap$ U $\subseteq$ U $\cap$ A.

   Theorem 2.4. A set A $\subseteq$ X *is M-collectionwise scattered*

in X *if and only if* A = $\cup N$ *where* N *is a well ordered*

subcollection of M, and for each M $\in N$ *there exists an open*

set U_M *such that* M $\subseteq U_M$ *and for all* K $\in N$, *if* K *does not pre-*

cede M *in* N, *then* K $\cap U_M \subseteq$ M.

   Proof. Assume A is M-collectionwise scattered, A $\neq \emptyset$,

and A = $\cup N$ as in Definition 2.1. Let N be well ordered by

the cardinal $|N|$ = λ. Let M_0 denote the first element of N

such that for some open U $\supseteq M_0$, K $\cap$ U $\subseteq M_0$ for all K $\in N$.

Let U_0 denote such an open set U. Assume M_β and U_β have been

defined for all $\beta < \alpha$ where $\alpha < \lambda$. Then by Definition 2.1

we may define M_α as the first element of N such that for

some open U $\supseteq M_\alpha$, if K $\in N \backslash \{M_\beta: \beta < \alpha\}$ then K $\cap$ U $\subseteq M_\alpha$. Let

U_α be such an open set. Thus sequences {M_α: $\alpha < \lambda$} and

$\{U_\alpha: \alpha < \lambda\}$ have been defined. If $M \in N$, then for some
$\alpha < \lambda$, $M = M_\alpha$. Thus $N = \{M_\alpha: \alpha < \lambda\}$. Thus N is expressed as
a well ordered set. If $M_\alpha \in N$, then for $\beta > \alpha$, $M_\beta \cap U_\alpha \subseteq M_\alpha$
by the construction.

Suppose the condition is satisfied by a well ordered
collection N. If $K \subseteq N$ and $K \neq \emptyset$, let M be the first element
of K. If $K \in K$ then K does not precede M. Hence $M \subseteq U_M$ and
$K \cap U_M \subseteq M$.

Definition 2.5. A subset A of a space X is σ-M-*collec-*
tionwise scattered in X if and only if $A = \cup\{S_n: n \in N\}$ where
each S_n is M-collectionwise scattered. If, in addition, each
S_n is closed in X, then A is called σ-*(closed and M-collec-*
tionwise scattered).

Terminology 2.6. If $A \subseteq X$ is M-collectionwise scattered
then by Definition 2.1 and Theorem 2.4 there exist a collec-
tion N and a collection $U = \{U_M: M \in N\}$ of open sets satisfy-
ing the conditions of Definition 2.1 or Theorem 2.4, respec-
tively. A pair $\langle N, U \rangle$ of such collections N and U will be
said to be an M-*pair for* A, respectively, an M-*pair in the*
order sense for A, if the conditions of 2.1, respectively
2.4, are satisfied by N and U.

Theorem 2.7. *Suppose* X *is a space and* A *is* σ-R-*collec-*
tionwise scattered in X *where* R *is a disjoint collection of*
σ-M-*collectionwise scattered subsets of* X. *Then* A *is*
σ-M-*collectionwise scattered in* X.

Proof. Suppose first that A is R-collectionwise scat-
tered. Let $\langle N, U(N) \rangle$ be an R-pair for A (see 2.6). For

each $M \in N$, $M = \cup\{S(M,n): n \in \omega\}$ and for each $n \in \omega$, $S(M,n)$

has an M-pair $\langle M'(M,n), U(M,n)\rangle$. Let $K_n = \{K:$ for some

$M \in N$, $K \in M'(M,n)\}$. Suppose $L \subseteq K_n$ and $L \neq \emptyset$. Then

$A = \{M \in N:$ for some $K \in L$, $K \in M'(M,n)\} \neq \emptyset$. Hence there

exists $A \in A$ and $U_A \supseteq A$ such that U_A is open and $L \cap U_A \subseteq A$

for all $L \in A$. Since R is a disjoint collection, $L \cap U_A = \emptyset$

if $L \neq A$. Let $W = \{K \in M'(A,n): K \in L\}$. Then $W \neq \emptyset$, so there

exist $B \in W$ and an open set $V_B \in U(A,n)$ such that $B \subseteq V_B$ and

$K \cap V_B \subseteq B$ for all $K \in W$. Suppose $K \in L$. If $K \in M'(A,n)$,

$K \cap V_B \cap U_A \subseteq B \cap U_A = B$, since $B \subseteq A$. If $K \in M'(L,n)$ for

some $L \in A$ such that $L \neq A$, then $K \cap V_B \cap U_A \subseteq V_B \cap L \cap U_A =$

$\emptyset \subseteq B$. Hence for each nonempty $L \subseteq K_n$ there exist $B \in L$ and

an open set $W_B \supseteq B$ such that $L \cap W_B \subseteq B$ for all $L \in L$. Hence

$T_n = \cup K_n$ is M-collectionwise scattered. Then $A = \cup\{T_n: n \in \omega\}$

is σ-M-collectionwise scattered. If A is σ-R-collectionwise

scattered, the preceding argument shows that A is the counta-

ble union of σ-M-collectionwise scattered sets.

   Theorem 2.8.   *If* X *is a space and* A $\subseteq$ X *is* R-*collection-*

wise scattered in X *and* R *is a disjoint collection of* M-*col-*

lectionwise scattered subsets of X, *then* A *is* M-*collection-*

wise scattered.

   Proof.   Similar to the preceding theorem.

   Theorem 2.9.   *Suppose* X *is a space,* A *is* M-*collection-*

wise scattered in X *(σ-M-collectionwise scattered in* X*) and*

B $\subseteq$ X *is such that* B $\cap$ M $\in$ M *for all* M $\in$ M. *Then* B $\cap$ A *is*

M-*collectionwise scattered in* X *(σ-M-collectionwise scattered*

in X*).*

   Proof.   Suppose $\langle N, U \rangle$ is an M-pair for A.   Let

$N|B = \{M \cap B: M \in N\}$. Then $B \cap A = \cup N|B$. If $K \neq \emptyset$ and $K \subseteq N|B$, let $K' = \{M \in N: M \cap B \in K\}$. Then $K' \neq \emptyset$ so there exists $M \in K'$ and an open $U_M \supseteq M$ such that for all $K \in K'$, $K \cap U_M \subseteq M$. Then $B \cap M \in K$ and if $B \cap K \in K$ where $K \in N$, then $K \in K'$, so $B \cap K \cap U_M \subseteq B \cap M$. Thus the first statement is proved. If A is a countable union of M-collectionwise scattered sets S_n, then each $B \cap S_n$ is M-collectionwise scattered.

Theorem 2.10. *Suppose* X *is a space and* A $\subseteq$ X *is covered by a collection* V *of open sets such that for all* V $\in$ V, V $\cap$ A *is* σ-M-*collectionwise scattered* (M-*collectionwise scattered*). *Suppose that for every closed* F $\subseteq$ X, F $\cap$ M $\in$ M *for all* M $\in$ M. *Then* A *is* σ-M-*collectionwise scattered* (M-*collectionwise scattered*).

Proof. Well order V and let $V' = \{V \in V: p(V,V) \cap A \neq \emptyset\}$ where $p(V,V) = V\backslash\cup\{V': V' \text{ precedes } V \text{ in } V\}$. Let $N = \{p(V,V) \cap A: V \in V\}\backslash\{\emptyset\}$. Then N is a disjoint collection of sets which are (σ)-M-collectionwise scattered by the preceding theorem. Theorem 2.7 (2.8) implies that A is σ-M-collectionwise scattered (M-collectionwise scattered).

Corollary 2.11. *If* X *is locally* M-*collectionwise scattered* (σ-M-*collectionwise scattered*) *where* M *is relatively closed hereditary* (cf. Theorem 2.10), *then* X *is* M-*collectionwise scattered* (σ-M-*collectionwise scattered*).

If X has property P locally and P is a closed hereditary property, then X is collectionwise M-scattered, where M is a disjoint collection of sets each of which has property P.

3. Primitive Sequences

We recall here some primitive sequence theory [WW_2, WW_3, WW_6] in order to facilitate the proofs of the main results.

Definition 3.1. Let $(Z, \leq)$ be a well ordered collection of sets. For each $W \in A$, let $p(w, A)$ denote $W \setminus \cup \{W' \in Z:$ $W' < W\}$. The set $p(w, Z)$ is called the *primitive part of* W *(in Z)*.

For $x \in \cup Z$, we let $F(x, Z)$ denote the first element of Z that contains x. $F(A, Z)$ is similarly defined for $A \subseteq W \in Z$.

Definition 3.2. A *primitive sequence H of A in X* is a sequence $H = \langle H_n: n \in N \rangle$ of well ordered subcollections of $P(X)$ such that for all $n \in N$:

(1) For all $H \in H_n$, $A \cap p(H, H_n) \neq \emptyset$.

(2) For all $x \in A$, $F(x, H_{n+1}) \subseteq F(x, H_n)$.

An *open primitive sequence* is a primitive sequence relative to a topological space X whose terms are well ordered collections of sets open in X.

Definition 3.3. Let H be a primitive sequence of A in X. A *primitive representative* of H is a sequence H = $\langle H_n: n \in N \rangle$ such that for all $n \in N$, $p(H_n, H_n) \cap p(H_{n+1}, H_{n+1}) \neq \emptyset$. The collection of all primitive representatives of H will be denoted by PR(H).

If $G = \langle G_n: n \in N \rangle$ is a sequence of collections of sets a *decreasing representative* of G is a sequence G such that $G_{n+1} \subseteq G_n \in G_n$ for all $n \in N$. The set of all decreasing representatives of G will be denoted by DR(G).

If H = $\langle H_n: n \in N \rangle$ and G = $\langle G_n: n \in N \rangle$ are decreasing

sequences, H is said to *dominate* G (or G *refines* H) if and
only if for all $n \in N$ there is $j > n$ such that $G_j \subseteq H_n$.

4. Scattering and Primitive Sequences

We present theorems here guaranteeing the existence of
primitive sequences for sets which are scattered with respect
to collections of sets having primitive sequences. These
theorems underlie the proofs of the results of section 5.

Theorem 4.1. Suppose X *is a space,* A *is* $\mathcal{M}$-*collectionwise
scattered in* X *and* $\langle \mathcal{N}, \mathcal{U} \rangle$ *is an* $\mathcal{M}$-*pair for* A *in the order
sense such that each* $M \in \mathcal{N}$ *has an open primitive sequence*
$V(M)$ *in* U_M. *Then there exists an open primitive sequence* $\mathcal{H}$
of A *in* X *such that for all* $H \in DR(\mathcal{H})$, *there exist* $k \in N$ *and*
$M \in \mathcal{N}$ *such that* $H_j = V_j$ *for all* $j \geq k$ *for some* $V \in DR(V(M))$.

Proof. Using the notation in the statement, let $\mathcal{H}_n =$
$\{H:$ for some $M \in \mathcal{N}$, $H \in V(M)_n$ and $p(M, \mathcal{N}) \cap p(H, V(M)_n) \neq \emptyset \}$.
If $H \in \mathcal{H}_n$ there is only one M associated with H as indicated.
For if $p(M, \mathcal{N}) \cap p(H, V(M)_n) \neq \emptyset$, and $p(M', \mathcal{N}) \cap p(H, V(M')_n) \neq \emptyset$
and $H \in V(M)_n$, then $H \subseteq U_M$. Hence $H \cap M' \subseteq M$ by Theorem 2.4.
Since $H \cap p(M', \mathcal{N}) \neq \emptyset$, it follows that M' does not follow M.
Similarly, M does not follow M' so $M = M'$. Temporarily denote
such an M by M(H). For H and H' in $\mathcal{H}_n$ define $H <_n H'$ if and
only if M(H) precedes M(H') in $\mathcal{N}$ or $M(H) = M(H')$ and H pre-
cedes H' in $V(M(H))_n$. It may be readily verified that $\leq_n$
well orders $\mathcal{H}_n$ and $\mathcal{H}_n$ covers A. Moreover, if $x \in A$ and
$M = F(x, \mathcal{N})$, then $F(x, \mathcal{H}_n) = F(x, V(M)_n)$. From this it is easy
to verify that $\mathcal{H} = \langle \mathcal{H}_n: n \in N \rangle$ is an open primitive sequence
of A in X. Suppose $H \in DR(\mathcal{H})$. Then for each $n \in N$, $H_n \in$
$V(M_n)_n$, where $M_n = M(H_n)$. In the well ordering of $\mathcal{N}$, let M_k

be the first element of the set of M_n's. If $j > k$, there

exists $x_j \in p(H_j, V(M_j)_j) \cap p(M_j, N)$. Since $x_j \in H_j \subseteq H_k$,

$x_j \in M_j \cap U_{M_k} \subseteq M_k$. Thus $M_k = M_j$. Hence $H_j \in V(M_k)_j$ for all

$j \geq k$. Let $V_j = H_j$ for $j \geq k$ and define V_j inductively for

$j < k$, by $V_j = F(V_{j+1}, V(M_k)_j)$. Then $V \in DR(V(M_k))$.

Remark 4.2. If A is M-collectionwise scattered in X and

each $M \in M$ has an open primitive sequence $V(M)$ in X, then

there exists an open primitive sequence H of A in X and for

each $M \in M$ an open set $U_M \supseteq M$ such that for all $H \in DR(H)$,

there exist $V \in DR(V(M)|U_M)$ and $k \in N$ such that $V_j = H_j$ for

all $j \geq k$.

Theorem 4.3. *Suppose* X *is a space,* A *is* M-*collectionwise*

scattered in X *and each* $M \in M$ *has an open primitive sequence*

V(M) *in* X. *Then* A *has an open primitive sequence* H *in* X *such*

that for all $H \in DR(H)$, *there exist* M *and* $V \in DR(V(M))$ *such*

that V *dominates* H.

Proof. Use the notation of Theorem 4.1, with the excep-

tion that V(M) need no longer be in U_M. Define $K(M)_n =$

$V(M)_n | U_M = \{V \cap U_M : V \in V(M)_n\}$ and note that K ↦ first element of

$\{V \in V(M)_n : V \cap U_M = K\}$ is an injection of $K(M)_n$ into $V(M)_n$.

Using the order on $K(M)_n$ induced by this mapping, it is

easily seen that $\langle K(M)_n : n \in N \rangle$ is a primitive sequence

of M in U_M. Now Theorem 4.1 applies and we obtain an open

primitive sequence H relative to the K(M) sequences. If

$H \in DR(H)$ there is M and $K \in DR(K(M))$ with the property indi-

cated in Theorem 4.1. Hence there exists $L \in PR(K(M))$ that

dominates K and hence H (take L_n = first element of $K(M)_n$

that includes a term of K). It is also easy to see that

there is $V \in PR(V(M))$ that dominates L. Hence the result.

The following simple lemmas are used to abbreviate the proof of the next theorem.

Lemma 4.4. Suppose $Z_1, \cdots, Z_k$ are well ordered collections of sets. Let $r_k(M_1, \cdots, M_k)$ denote the relation $M_i \in Z_i$ for $i = 1, \cdots, k$ and $\cap\{p(M_i, Z_i): i \leq k\} \neq \emptyset$. Then if $M_1 \cap \cdots \cap M_k = M_1' \cap \cdots \cap M_k'$ where $M_i, M_i' \in Z_i$ for all $i \leq k$ and $r_k(M_1, \cdots, M_k)$ and $r_k(M_1', \cdots, M_k')$ hold, then $M_i = M_i'$ for all $i \leq k$.

Proof. Since $\cap\{p(M_i, Z_i): i \leq k\} \neq \emptyset$ and $\cap\{p(M_i', Z_i): i \leq k\} \neq \emptyset$, it follows that for each $i \leq k$, $M_i \cap p(M_i', Z_i) \neq \emptyset \neq M_i' \cap p(M_i, Z_i)$. Hence $M_i = M_i'$ for all $i \leq k$.

Lemma 4.5. Suppose $Z_1, \cdots, Z_k$ are well ordered collections of sets. Then the set $W = \{M_1 \cap \cdots \cap M_k: r_k(M_1, \cdots, M_k)\}$ has a well ordering determined by the orderings on the Z_i such that for all $x \in \cup W$,

$$F(x, W) = F(x, Z_1) \cap \cdots \cap F(x, Z_k).$$

Proof. By the preceding lemma, the expression $M_1 \cap \cdots \cap M_k$ is unique for elements of W. For $W, W' \in W$, define $W < W'$ if and only if $W \neq W'$ and if i is the first j such that $M_j \neq M_j'$, then M_i precedes M_i' in Z_i. The verification that $\leq$ well orders W is straightforward. Suppose $x \in W$ and $F(x, W) = M_1 \cap \cdots \cap M_k$. Since $F(x, Z_i) \leq M_i$ (in Z_i), it follows that $M_i = F(x, Z_i)$.

Remark 4.6. We will refer to well orderings given by Lemma 4.5 as "natural well orderings." Note that in the application of Lemma 4.5 in the proof below some of the sets

Z_i consist of a single element; we consider such a Z_i as a
well ordered set of one element. We will well order finite
unions of disjoint well ordered sets below in the standard
way: If $W = U_1 \cup \cdots \cup U_n$, the elements of U_i precede those
of U_j for i < j and the elements of each U_i retain the order
of U_i.

   Theorem 4.7. Suppose X is a space, $A \subseteq X$, and A =
$\cup\{S_n: n \in N\}$ *where each* S_n *is closed and has an open primitive*
sequence $W^n = \langle W_m^n: m \in N \rangle$ *in X. Then A has an open primi-*
tive sequence H *in X such that if* $G \in DR(H)$ *and* $\cap\{G_n:$
$n \in N\} \neq \emptyset$, *then there exist* $k \in N$ *and* $W \in PR(W^k)$ *such that*
W dominates G.

   Proof. Define $F_0 = \emptyset$ and $F_n = S_1 \cup \cdots \cup S_n$. Define
$V_m^k = \{W \backslash F_{k-1}: W \in W_m^k \text{ and } p(W, W_m^k) \cap (S_k \backslash F_{k-1}) \neq \emptyset\}$. Lemma
4.5 applies, so we give V_m^k the natural well ordering. If
V_1^1 covers A, define J_1 as $\emptyset$, otherwise let $J_1 = \{X \backslash S_1\}$.
Define $H_1 = V_1^1 \cup J_1$ and well order it as a well ordered
union. Then H_1 is critically well ordered (i.e. well ordered
and $p(H, H_1) \neq \emptyset$ for all $H \in H_1$) and covers A. Assume
$H_1, \cdots, H_{n-1}$ have been defined and are critically well ordered
open covers of A.

   For $k \leq n$ and $m \leq n$ and $k + m = n + 1$ define
$B_m^k = \{V \cap H: V \in V_m^k, H \in H_{n-1}, \text{ and } p(V, V_m^k) \cap p(H, H_{n-1}) \cap$
$(S_k \backslash F_{k-1}) \neq \emptyset\}$. Give each B_m^k the natural well ordering and
let $H_n' = B_1^n \cup B_2^{n-1} \cup \cdots \cup B_n^1$ well ordered as a disjoint
union. Let $J_n = \{H \backslash F_n: H \in H_{n-1} \text{ and } p(H, H_{n-1}) \cap A \backslash F_n \not\subseteq \cup H_n'\}$
with the natural well ordering.

   Let $H_n = H_n' \cup J_n$ well ordered as a disjoint union. Then

$\mathcal{H}_n$ is a critically well ordered open cover of A and it may be
verified that $\mathcal{H} = \langle \mathcal{H}_n : n \in N \rangle$ is an open primitive sequence
of A in X.

Suppose $G \in DR(\mathcal{H})$ and $\cap\{G_n : n \in N\} \neq \emptyset$. Let k be the
first integer j such that for some $x \in \cap\{G_n : n \in N\}$, $x \in S_j$.
Then $x \in S_k \backslash F_{k-1}$ and $x \notin \cup \mathcal{V}_m^i$ for any $i > k$. Moreover $x \notin \cup \mathcal{I}_n$
for $n > k$. Thus for $n > k$ there exists $j \leq k$ such that
$G_n \in \mathcal{B}_{n-j+1}^j$. Since $G_n \in \mathcal{B}_m^j$ implies that $G_{n+1} \notin \mathcal{B}_{m'}^{j'}$ for $j' < j$,
it follows that for some $m \geq k$ there exists $t \leq k$ such that
$G_n \in \mathcal{B}_{n-t+1}^t$ for all $n \geq m$. From this it follows that for each
$n \in N$ there exists W_n such that W_n is the first element of
$\mathcal{W}_n^t$ that includes some term of G. By a standard argument
(Lemma 2.2 of [WW$_2$]) it follows that $W \in PR(\mathcal{W}^t)$.

Theorem 4.8. *Let X be a space, and suppose that A is*
σ-(closed and $\mathcal{M}$-collectionwise scattered) in X and each
$M \in \mathcal{M}$ has an open primitive sequence $\mathcal{V}(M)$ in X. Then there
exists an open primitive sequence $\mathcal{H}$ of A in X such that for
all $H \in DR(\mathcal{H})$, if $\cap\{H_n : n \in N\} \neq \emptyset$, then there exist M and
$V \in PR(\mathcal{V}(M))$ such that V dominates H.

Proof. The set $A = \cup\{S_n : n \in N\}$, where each S_n is
closed and $\mathcal{M}$-collectionwise scattered. By Theorem 4.3, each
S_n has an open primitive sequence $\mathcal{W}^n$ such that for each
$W \in DR(\mathcal{W}^n)$ there is M and $V \in PR(\mathcal{V}(M))$ such that V dominates
W. By Theorem 4.7, A has an open primitive sequence $\mathcal{H}$ in X
such that for all $H \in DR(\mathcal{H})$ if $\cap\{H_n : n \in N\} \neq \emptyset$, then there
exist n and $W \in PR(\mathcal{W}^n)$ such that W dominates H. Thus there
exist M and $V \in PR(\mathcal{V}(M))$ such that V dominates W and, hence
V dominates H.

Theorem 4.9. *Let* X *be a space and suppose that* A *is* σ-$\mathcal{M}$-*collectionwise scattered in* X *and each* M $\in$ $\mathcal{M}$ *has an open primitive sequence* $\mathcal{V}$(M) *in* X. *Then there exists an open primitive sequence* $\mathcal{H}$ *of* A *in* X *such that for all* H $\in$ PR($\mathcal{H}$), *if* pc(H) = $\cap\{p(H_n,\mathcal{H}_n) n \in N\} \neq \emptyset$, *then there exist* M *and* V $\in$ PR($\mathcal{V}$(M)) *such that* V *dominates* H *and* pc(H) $\subseteq$ pc(V).

Proof. The proof is similar to, but simpler than that of Theorem 4.7. A is the union $\cup\{S_n : n \in N\}$ of $\mathcal{M}$-collectionwise scattered sets. Each S_n has an open primitive sequence $\mathcal{W}^n$ formed as in Theorem 4.3. Let $\mathcal{H}_1 = \mathcal{W}^1_1$ with X added as a final element if $\mathcal{W}^1_1$ does not cover A. Assuming $\mathcal{H}_1, \cdots, \mathcal{H}_{n-1}$ to be critically well ordered sets covering A, for k $\leq$ n and m $\leq$ n and k + m = n + 1, define β^k_m = {V $\cap$ H: V $\in$ $\mathcal{W}^k_m$, H $\in$ $\mathcal{H}_{n-1}$, and p(V,$\mathcal{W}^k_m$) $\cap$ p(H_{n-1}) $\cap$ ($S_k \setminus F_{k-1}$) $\neq$ $\emptyset$}, where F_0 = $\emptyset$ and F_n = $S_1 \cup \cdots \cup S_n$. Let $\mathcal{H}_n = \beta^n_1 \cup \beta^{n-1}_2 \cup \cdots \cup \beta^1_n$ with the natural well ordering (together with X as a final element if $\mathcal{H}_n$ does not cover A). It may be verified that $\mathcal{H}$ is an open primitive sequence which satisfies the condition.

5. Base of Countable Order Theory

We briefly summarize some concepts used in the theorems below, and prove three further theorems useful in the proofs.

Suppose H is a sequence. If $\{H_n : n \in N\}$ consists of open sets which are a base at a point x of a space, we say H *is a base at* x.

We list the following properties of decreasing sequences B = $\langle B_n : n \in N \rangle$ of open sets in a space X and A $\subseteq$ X (cf. [CČN] and [W_4]):

Definition 5.1.

(λ)   There exists $x \in X$ such that every open set con-
        taining x includes some B_n and B is a base at all
        elements of $\cap\{B_n: n \in N\}$.

(b)   B is a base at all elements of $\cap\{B_n: n \in N\}$.

(p)   If $\mathcal{J}$ is a filterbase in X and $\mathcal{J}$ refines B (i.e. each
        B_n includes some $F \in \mathcal{J}$), then $\cap\{\overline{F}: F \in \mathcal{J}\} \neq \emptyset$.

(q)   If $\mathcal{J}$ is a countable filterbase in X and $\mathcal{J}$ refines
        B, then $\cap\{\overline{F}: F \in \mathcal{J}\} \neq \emptyset$.

(d) $\cap\{B_n: n \in N\}$ is empty or a singleton.

(sA) $\cap\{B_n: n \in N\} \subseteq A$.

Remark 5.2.   Note that all properties listed are *mono-*
tonic [CČN]:   A property (M) of decreasing sequences B is
monotonic if and only if for every decreasing sequence W of
open sets, if B dominates W and B has (M), then W has (M).
This fact, together with the nature of the definitions be-
low, permits the theorems below to be derived from those of
section 4.

Now we define certain types of open primitive sequences.
Suppose $\mathcal{W}$ is an open primitive sequence of A in X.   Let

$PR(\mathcal{W},A) = \{W \in PR(\mathcal{W}): p(W_n,\mathcal{W}_n) \cap A \neq \emptyset$ for all
            $n \in N\}$,

$PR^{\mu}(\mathcal{W},A) = \{W \in PR(\mathcal{W},A): A \cap \cap\{W_n: n \in N\} \neq \emptyset\}$,

$PR^{\pi}(\mathcal{W},A) = \{W \in PR(\mathcal{W},A): A \cap \cap\{p(W_n,\mathcal{W}_n): n \in N\} \neq \emptyset\}$.

Definition 5.3.   Let (c) denote any of the properties
listed in Definition 5.1.   Then an open primitive sequence
$\mathcal{W}$ of A in X is called:

a (c)-*sequence of* A *in* X if and only if every

W $\in$ PR($\mathscr{W}$,A) has (c),

a (μc)-*sequence of* A *in* X if and only if every

W $\in$ PR$^\mu$($\mathscr{W}$,A) has (c),

a (πc)-*sequence of* A *in* X if and only if every

W $\in$ PR$^\pi$($\mathscr{W}$,A) has (c).

If A = X we speak of a (c)-sequence in X, etc.

Remark 5.4.   Note that if M has an (α)-sequence in X,
where (α) is any of the properties listed in the preceding
definition, then if F $\subseteq$ X is closed, F $\cap$ M has an (α)-sequence
in X.

The following theorem summarizes theorems and definitions
of [WW$_3$, WW$_2$] and the present paper.

Theorem 5.5.   *An essentially* T$_1$ *space has the property*
listed below on the left if and only if it has an open
primitive (α)-*sequence where* (α) *is listed at right.*

λ-*base*	(λ)
base of countable order	(μb)
primitive base	(πb)
primitively quasi-complete *(or primitively q-space)*	(πq)
primitively p-*space*	(πp)
primitive diagonal	(πd)
diagonal a set of interior condensation	(μd).
If X *is pararegular* [WW$_6$],	
β_b-*space*	(μp)
β_c-*space*	(μq)
λ_b-*space*	(p)

λ_c-*space*   (q)

If $A \subseteq X$ *then* A *has the property*

set of interior condensation   (μsA).

We add the following definitions as natural accompani-
ments to the foregoing (see Remark 5.9).

Definition 5.6. A space X is said to be:

a *monotonically* p-*space* if and only if it has a (μp)-sequence;

a *monotonically* q-*space* if and only if it has a (μq)-sequence.

A space X is said to be:

(p)-*complete* if and only if it has a (p)-sequence;

(q)-*complete* if and only if it has a (q)-sequence.

The above concepts coincide, respectively, with β_b, β_c,

λ_b, λ_c-spaces [W$_1$], when X is pararegular [WW$_6$].

The following definition is useful in stating the main

results.

Definition 5.7. Let X be a space and $M \subseteq X$. Let P be
any of the properties listed in Theorem 5.5 or Definition 5.6
defined by some property (α). We say that M *has* P *in the*
global topology if and only if there is an (α)-sequence of M
in X.

Remarks 5.8.

Thus for example, M has a base of countable order in the
global topology if and only if there exists a sequence
$\mathcal{G} = \langle \mathcal{G}_n : n \in N \rangle$ of open sets such that each $\mathcal{G}_n$ is a basis
for the topology of X at all points of M and if G is a de-
creasing representative of $\mathcal{G}$ such that $x \in \cap\{G_n : n \in N\} \cap M$,

then $\{G_n : n \in N\}$ is a base at x.

    If X is a first countable space then each $\{x\} \subseteq X$ has

a base of countable order (in fact a strong λ-base (see 6))

in the global topology. If X is of point countable type

(respectively, a q-space), then each $\{x\} \subseteq X$ is (p)-complete,

((q)-complete) in the global topology.

    We note that the concepts of primitive base, primitively

quasi-complete, and primitively p-space can be formulated

more simply in terms of a sequence $\mathscr{W}$ of well ordered open

covers of X by requiring an appropriate condition on

$\{F(x, \mathscr{W}_n) : n \in N\}$ for all $x \in X$.

    Thus X has a primitive base if and only if there exists

a sequence as described such that for all $x \in X$, $\{F(x, \mathscr{W}_n) :$

$n \in N\}$ is a base at x.

    Remarks 5.9.

    The following remarks are intended to help in organizing

certain of the previously named concepts. Consider the four

properties: (1) space of point countable type $[A_1]$, (2)

q-space [M], (3) first countable space, (4) points are G_δ's.

The monotonic versions of these properties are, respectively:

(M1) monotonically p-space, (M2) monotonically q-space, (M3)

spaces having bases of countable order, (M4) diagonal a set

of interior condensation. The primitive versions are, re-

spectively: (p1) primitively p-space, (p2) primitively q-

space, (p3) primitive base, (p4) primitive diagonal. Thus

some of the most frequently used monotone and primitive spaces

are uniformizations of first countability or some of its

generalizations.

In the following theorems let X be a space, $A \subseteq X$, and M a collection of subsets of X.

Theorem 5.10. *Suppose A is* M-*collectionwise scattered in X and each* $M \in M$ *has an* (α)-*sequence in X where* (α) *is any of the properties of the right hand list in Theorem 5.5. Then each* $M \in M$ *has an* (α)-*sequence* V(M) *and A has an* (α)-*sequence* H *such that for all* $H \in DR(H)$ *there is some* $M \in M$ *and* $k \in N$ *such that* $H_j = V_j$ *for all* $j \geq k$ *and some* $V \in DR(V$(M)).

Proof. If $\langle N, U \rangle$ is an M-pair for A in the order sense, we can, using the technique of proof of Theorem 4.7, obtain an (α)-sequence V(M) of M in U_M for all $M \in N$. By Theorem 4.1, there is an open primitive sequence H of A in X such that each $H \in DR(H)$ satisfies the stated condition. Since each V(M) is an (α)-sequence, each $H \in DR(H)$ will have (α), because the $V \in DR(V$(M)) given by the theorem has (α).

Theorem 5.11. *Suppose A is* σ-*(closed and* M-*collectionwise scattered) in X and each* $M \in M$ *has an* (α)-*sequence in X, where* (α) *is any of the properties of Theorem 5.5 involving* μ *or* π. *Then A has an* (α)-*sequence in X.*

Proof. Let V(M) be an (α)-sequence for each $M \in M$. There exists an open primitive sequence H of A in X satisfying the conclusion of Theorem 4.8. Suppose $H \in DR(H)$ and $\cap\{H_n: n \in N\} \neq \emptyset$. Then if $V \in PR(V$(M)) dominates H, $V \in PR(H)$. Thus if (α) = (μc) or (πc), V has (c) and thus by Remark 5.2, so does H.

Theorem 5.12. *If A is* σ-M-*collectionwise scattered and*

each M $\in$ $\mathcal{M}$ *has an* (α)-*sequence where* (α) *is a property*

listed in 5.3 involving π, *then* A *has an* (α)-*sequence*.

 Proof. Each M $\in$ $\mathcal{M}$ has an (α)-sequence $\mathcal{V}$(M) in X. Hence

there is an open primitive sequence $\mathcal{H}$ of A in X satisfying

the condition of Theorem 4.9. If H $\in$ PR$^{\pi}$($\mathcal{H}$), then there is

M and V $\in$ PR($\mathcal{V}$(M)) such that V dominates H. Since pc(V) $\neq$ $\emptyset$,

it follows that if $\mathcal{V}$(M) has (πc) then V has (c) and so does

H. Hence $\mathcal{H}$ has (πc).

6. Main Results

 In the theorems below, we assume that the set $\mathcal{M}$ is a

relatively closed hereditary subcollection of the power set

of a space X. This is justified in view of Remark 5.4. See

5.7 for explanation of the usage of *global topology*.

 We say that X has a *strong* λ-*base* if and only if X has

a (λ)-sequence $\mathcal{H}$ such that if H $\in$ PR($\mathcal{H}$), then $\cap\{H_n: n \in N\} \neq \emptyset$.

A set M $\subseteq$ X has a *strong* λ-*base in the global topology* pro-

vided it has a (λ)-sequence $\mathcal{H}$ in X such that for all H $\in$ PR($\mathcal{H}$),

M $\cap$ $\cap\{H_n: n \in N\} \neq \emptyset$.

 Theorem 6.1. *Suppose* X *is a locally* $\mathcal{M}$-*collectionwise*

scattered space.

 (a) *If each* M $\in$ $\mathcal{M}$ *has a strong* λ-*base* (λ-*base*) *in the*

global topology, then X *has a strong* λ-*base* (λ-*base*).

 (b) *If each* M $\in$ $\mathcal{M}$ *is* (p)-*complete* ((q)-*complete*) *in the*

global topology, then X *is* (p)-*complete* ((q)-*complete*). *If*

X *is pararegular and each* M $\in$ $\mathcal{M}$ *is a* λ_b-(λ_c)-*space, then* X

is a λ_b-(λ_c)-*space.*

 Proof. (a) By Corollary 2.1 and our blanket assumption

on $\mathcal{M}$, X is $\mathcal{M}$-collectionwise scattered. Each M $\in$ $\mathcal{M}$ has a

(λ)-sequence (respectively, a strong λ-sequence). By Theorem
5.10, X has the same kind of sequence. The proof of part (b)
is similar.

   Corollary 6.2. *Suppose* X *is a locally scattered* T_1-*space.*
If X *is first countable, then* X *has a strong* λ-*base. If* X
is T_2 *of point countable type, then* X *is* (p)-*complete. If*
X *is a* q-*space, then* X *is* (q)-*complete.*

   Proof. Here $M = \{\{x\}: x \in X\}$. By Remarks 5.8 the
hypotheses of the theorem are satisfied.

   Theorem 6.3. *Suppose* X *is locally* σ-(*closed in* X *and*
M-*collectionwise scattered*).

   (a) *If each* M $\in$ M *has a base of countable order in the*
global topology, then X *has a base of countable order.*

   (b) *If each* M $\in$ M *is a monotonically* p-*space* (*respectively,*
a monotonically q-*space*) *in the global topology, then* X *is a*
monotonically p-*space* (*respectively, a monotonically* q-*space*).
If X *is pararegular and each* M $\in$ M *is a* β_b-*space* (β_c-*space*)
in the global topology, then X *is a* β_b-*space* (β_c-*space*).

   (c) *If each* M $\in$ M *has diagonal a set of interior con-*
densation, then X *also has such a diagonal.*

   Proof. (a) Suppose $U \subseteq X$ is open and σ-(closed in X and
M-collectionwise scattered). Since each M $\in$ M has a base of
countable order in the global topology, each M $\in$ M has a
(μb)-sequence in X. By Theorem 5.11, U has a (μb)-sequence
in X. The conclusion follows from the local-implies-global
property of base of countable order [WoW]. It may also be
deduced from Corollary 2.11 of the present paper.

   The proofs of (b) and (c) are similar.

Corollary 6.4. *Suppose* X *is a locally* σ-*(closed in* X *and scattered)* T_1 *space.*

(a) *If* X *is first countable, then* X *has a base of countable order.*

(b) *If* X *is* T_2 *and of point countable type (a* q-*space) then* X *is a monotonically* p-*space (*q-*space).*

(c) *If points are* G_δ's *in* X, *then* X *has diagonal a set of interior condensation.*

Proof. Here again take $\mathcal{M} = \{\{x\}: x \in X\}$. The conditions stated then translate into the corresponding conditions of the preceding theorem for the sets $M = \{x\} \in \mathcal{M}$.

Theorem 6.5. *Suppose* X *is a space and* $A \subseteq X$. *If* A *is* σ-*(closed and* $\mathcal{M}$-*collectionwise scattered) in* X *where each* $M \in \mathcal{M}$ *is a set of interior condensation, then* A *is a set of interior condensation.*

Proof. Each $M \in \mathcal{M}$ has a (μsM)-sequence in X. By Theorem 5.11, A has a (μsA)-sequence in X, so that A is a set of interior condensation, by 5.3.

Theorem 6.6. *Suppose* X *is a locally* σ-$\mathcal{M}$-*collectionwise scattered space.*

(a) *If each* $M \in \mathcal{M}$ *has a primitive base in the global topology then* X *has a primitive base.*

(b) *If each* $M \in \mathcal{M}$ *is a primitively* p-*space (*q-*space) in the global topology, then* X *is a primitively* p-*space (*q-*space).*

(c) *If each* $M \in \mathcal{M}$ *has a primitive diagonal in the global topology, then* X *has a primitive diagonal.*

Proof. By Corollary 2.11, X is σ-$\mathcal{M}$-collectionwise

scattered. Since each M ∈ $\mathcal{M}$ has an appropriate type of
(α)-sequence in X as listed in Theorem 5.12, it follows that
X has such an (α)-sequence. Application of Theorem 5.3 com-
pletes the proof.

Corollary 6.7. Let X *be a locally σ-scattered space.*

(a) If X *is first countable, then* X *has a primitive*
base.

(b) If X *is* T_2 *and of point countable type (a q-space)*
then X *is a primitively* p-space *(q-space).*

(c) If points are G_δ *'s in* X, *then* X *has a primitive*
diagonal.

Proof. This follows from Theorem 6.6, analogously to
the way in which Corollary 6.4 follows from Theorem 6.3.

7. Applications and Examples

We make application here to the case of scattered and
σ-scattered spaces as illustrations.

Theorem 7.1. Let X *be a* T_1 *locally scattered* q-space.

(a) If points are G_δ *'s and* X *is hereditarily weakly*
θ-*refinable then* X *is quasi-developable and has a strong*
λ-*base.*

(b) If closed sets are G_δ *'s and* X *is weakly* θ-*refinable,*
then X *is developable and semi-complete.*

(c) If X *is collectionwise normal, closed sets are*
G_δ *'s, and* X *is weakly* θ-*refinable, then* X *is completely*
metrizable.

Proof. Since X is locally scattered, it is scattered.
Since X is a T_1, q-space and thus in (a), (b), and (c) points

are G_δ's, it follows that X is T_1 and first countable [L].
Hence by 6.2, X has a strong λ-base. Part (a) then follows
from [BB]. Part (b) follows from (a) and [BL]. Part (c)
follows from (b) and [B] and the fact that a metric space
with a λ-base is completely metrizable [WW_1].

   Theorem 7.2. Suppose X *is a regular locally σ-(closed
and scattered) space such that points are G_δ's. Then* X *has
a base of countable order if and only if* X *is a monotonically
p-space.*

   Proof. By 6.4 (c), X has diagonal a set of interior
condensation. A regular monotonically p-space with such a
diagonal has a base of countable order [W_3].

   Theorem 7.3. Let X *be a locally σ-(closed and scattered)
regular T_1 q-space. Then if* X *is θ-refinable, it is a
p-space.*

   Proof. By 6.4 (b), X is a monotonically q-space. By
[WW_5], it is a p-space.

   Theorem 7.4. Let X *be a scattered Tychonoff space of
point countable type. If* X *is θ-refinably embedded in* X,
then X *is Čech complete.*

   Proof. By 6.2, X is a λ_b-space. The result follows
from [WW_5].

   Theorem 7.5. Suppose X *is a locally countable T_1
q-space. Then* X *has a base of countable order.*

   Proof. Since X is locally countable, it is locally
σ-scattered, hence σ-scattered. Thus locally it has a primi-
tive base and closed sets are G_δ's. By [WW_4], locally it

has a base of countable order and thus has one globally [WoW].

   Theorem 7.6. *Suppose* X *is a locally* σ-*scattered* T_1
q-*space*. *If closed sets are sets of interior condensation in*
X, *then* X *has a base of countable order*.

   Proof. By 6.7 such a space is T_1 and has a primitive
base because the hypothesis implies first countability. By
[WW$_4$], the space has a base of countable order.

   That the classes of scattered, σ-scattered, σ-(closed and
scattered) first countable spaces are distinct is shown by the
following examples.

   Example 7.7. The space Q of rationals with the usual
topology is a σ-(closed and scattered) space which is not
scattered. It does not have a λ-base since it doesn't have
the Baire property.

   Example 7.8. The so-called Michael line [SS, p. 90] is
a σ-scattered first countable T_2 paracompact space which is
not σ-(closed and scattered). This follows from the fact
that it cannot have a base of countable order.

   Example 7.9. The space ω_1 with the order topology shows
that no stronger property such as developable or even quasi-
developable is implied by being scattered, normal, and first
countable.

References

[A$_1$]   A. V. Arhangel'skiĭ, *Bicompact sets and the topology*
        of spaces, Tr. Mosk. Mat. Obšč. 13 (1965), 3-55.

[BB]   H. R. Bennett and E. S. Berney, *On certain generaliza-*
        tions of developable spaces, Gen. Top. Appl. 4 (1974),
        43-50.

[BL]   H. R. Bennett and D. J. Lutzer, *A note on weak
        θ-refinability*, Gen. Top. Appl. 2 (1972), 49-54.

[B]   R. H. Bing, *Metrization of topological spaces*, Can.
        J. Math. 3 (1951), 175-186.

[CČN]   J. Chaber, M. M. Čoban and K. Nagami, *On monotonic
        generalizations of Moore spaces, Čech complete spaces,
        and p-spaces*, Fund. Math. 84 (1974), 107-119.

[K]   K. Kuratowski, *Topology* I, Academic Press, New York,
        1966.

[L]   D. J. Lutzer, *Semimetrizable and stratifiable spaces*,
        Gen. Top. Appl. 1 (1971), 43-48.

[M]   E. Michael, *A note on closed maps and compact sets*,
        Israel J. Math. 2 (1964), 173-176.

[SS]   L. A. Steen and J. A. Seebach, Jr., *Counter-examples
        in topology*, Holt, Rinehart, and Winston, New York,
        1970.

[T]   R. Telgársky, *Total paracompactness and paracompact
        dispersed spaces*, Bull. Acad. Pol. Sci. 16 (1968),
        567-572.

[W$_1$] H. H. Wicke, *Open continuous images of certain kinds
        of M-spaces and completeness of mappings and spaces*,
        Gen. Top. Appl. 1 (1971), 85-100.

[W$_2$] _____, *Base of countable order theory and some
        generalizations*, in Proc. Univ. of Houston Point Set
        Topology Conf. 1971, Houston, Tex., 1972, 76-95.

[W$_3$] _____, *On spaces whose diagonal is a set of in-
        terior condensation*, Notices Amer. Math. Soc. 19
        (1972), A-567.

[W$_4$] _____, *Complete mappings in base of countable order
        theory*, in Set Theoretic Topology, Academic Press,
        1977, 383-412.

[WW$_1$] _____ and J. M. Worrell, Jr., *Open continuous map-
        pings of spaces having bases of countable order*,
        Duke Math. J. 34 (1967), 255-272.

[WW$_2$] _____, *Topological completeness of first countable
        Hausdorff spaces* I, Fund. Math. 75 (1972), 209-222.

[WW$_3$] _____, *Primitive structures in general topology*,
        in Studies in Topology, Academic Press, New York,

1975, 581-599.

[WW$_4$] H. H. Wicke and J. M. Worrell, Jr., *A characterization of spaces having bases of countable order in terms of primitive bases*, Can. J. Math. 27 (1975), 1100-1109.

[WW$_5$] _____, *The concept of a θ-refinable embedding*, Gen. Top. Appl. 6 (1976), 167-181.

[WW$_6$] _____, *Topological completeness of first countable Hausdorff spaces* II, Fund. Math. 91 (1976), 11-27.

[WW$_7$] _____, *The local-implies-global characteristic of primitive sequences*, in Proceedings Memphis State University Topology Conference, Marcel Dekker, N.Y., 1976, 269-282.

[WW$_8$] _____, *Scattered spaces of point-countable type*, 1976 Prague Topology Conference, to appear.

[WoW] J. M. Worrell, Jr. and H. H. Wicke, *Characterizations of developable topological spaces*, Can. J. Math. 17 (1965), 820-830.

Ohio University

Athens, Ohio 45701