
TOPOLOGY PROCEEDINGS



Volume 2, 1977

Pages 281–307

<http://topology.auburn.edu/tp/>

SPACES WHICH ARE SCATTERED WITH RESPECT TO COLLECTIONS OF SETS

by

H. H. WICKE AND J.M. WORRELL, JR.

Topology Proceedings

Web: <http://topology.auburn.edu/tp/>

Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA

E-mail: topolog@auburn.edu

ISSN: 0146-4124

COPYRIGHT © by Topology Proceedings. All rights reserved.

SPACES WHICH ARE SCATTERED WITH RESPECT TO COLLECTIONS OF SETS

H. H. Wicke and J.M. Worrell, Jr.

We study in this paper some of the topological structure associated with scattered spaces, σ -scattered spaces, or, more generally, spaces which are collectionwise or σ -collectionwise scattered with respect to collections of sets which are either complete or satisfy some sort of generalized uniform first countability condition. A number of important counterexamples are spaces of this kind. One of the earliest results known to us of the type considered is that of Kuratowski [K]: Every separable metric scattered space is an absolute G_δ . Telgársky [T] proved that every T_2 paracompact first countable scattered space is an absolute metric G_δ . The authors [WW₂] showed that every T_1 first countable scattered space has a λ -base. (Both the results of Kuratowski and Telgársky follow from this theorem which is a point of departure for the present paper.) Subsequently they showed (announced in [WW₇]) that σ -(closed and scattered) T_1 first countable spaces have a base of countable order and σ -scattered first countable spaces have primitive bases. Furthermore, by replacing *first countable* by *point-countable type* or by *q-space*, analogues of the above results are obtained involving the β_b , β_c , λ_b , λ_c -spaces [W₁] and primitively quasi-complete spaces. These results will be obtained here as special cases of theorems involving generalizations of

scattered and σ -*scattered*. The main concept is that of a space being scattered with respect to a collection \mathcal{M} of its subsets (Definition 2.1). In this generalization, the points of scattered spaces are replaced by certain sets having suitable properties: for example, each $M \in \mathcal{M}$ is compact and of countable character or each $M \in \mathcal{M}$ has a base of countable order in the global topology. An analogue of the first theorem stated above is: Suppose X is \mathcal{M} -collectionwise scattered with respect to a collection \mathcal{M} of sets having a base of countable order (λ -base) in the global topology. Then X has a base of countable order (λ -base). The various local-implies-global theorems of base of countable order theory [WW₈] are special cases of the results obtained here.

Passing from *scattered* to *collectionwise scattered* greatly widens the scope of the results previously obtained. Simple examples show this: No uncountable separable metric connected space can be σ -scattered; however such a space can be σ - \mathcal{M} -collectionwise scattered, where \mathcal{M} is a disjoint collection of arcs.

We present first the main definition and some consequences. In order to derive a number of results efficiently, we then prove some theorems concerning spaces which are scattered with respect to collections of sets each of which has an open primitive sequence in the global topology. The results indicate that the primitive sequence theory is a natural setting for the discussion of collectionwise scattered spaces. We then apply these theorems to get results concerning some of the concepts of base of countable order theory including those mentioned above. Examples show that these results cannot be

improved, say, by requiring collectionwise scattered with respect to sets which are metrizable in the global topology.

Terminology. A space is *scattered* if and only if it contains no nonempty dense-in-itself subset $[K]$; it is called σ -*scattered* if it is the union of a countable collection of scattered subsets; if each of these scattered subsets is closed, the space is called σ -(*closed and scattered*).

As an heuristic guide, we emphasize that for first countable T_1 spaces, scattered implies a strongly complete base of countable order $[WW_2]$, σ -(closed and scattered) implies base of countable order $[WW_8]$, and σ -scattered implies primitive base $[WW_8]$. (Proofs of these statements are given in Section 7 below.) Analogues of these results hold for collectionwise scattered spaces and this is the underlying organizing principle of the presentation; see especially Sections 4 and 5. The replacement of points by spaces having certain structures in the global topology leads to Theorems 6.1, 6.3, and 6.6 which are generalizations of the theorems just quoted (in the order as given). We note that a number of the results of Section 7 were presented at the Prague Conference $[WW_8]$; a summary is to appear in the Proceedings of the Conference.

2. Collectionwise Scattered Sets

The definition of collectionwise scattered sets is given here and various properties are established.

Definition 2.1. Let X be a space, $A \subseteq X$ and \mathcal{A} a collection of subsets of X . Then A is called \mathcal{A} -*collectionwise*

scattered in X if and only if there exists $N \subseteq \mathcal{M}$ such that $A = \bigcup N$ and for all $K \subseteq N$, if $K \neq \emptyset$, there exist $M \in K$ and an open set U_M such that $M \subseteq U_M$ and for all $K \in K$, $K \cap U_M \subseteq M$.

Theorem 2.2. A set $A \subseteq X$ is scattered in the usual sense if and only if it is \mathcal{M} -collectionwise scattered, where $\mathcal{M} = \{\{x\}: x \in A\}$.

Theorem 2.3. Suppose $A \subseteq X$ has property P locally (i.e. for all $x \in A$ there is an open U in X such that $x \in U$ and $U \cap A$ has P). Then if \mathcal{M} is the set of all subsets of X having P , the set A is \mathcal{M} -collectionwise scattered in X .

Proof. Let $\mathcal{N} = \{U \cap A: U \text{ is open in } X \text{ and } U \cap A \text{ has } P\}$. Then $A = \bigcup \mathcal{N}$. If $U \cap A, V \cap A \in \mathcal{N}$ then $U \cap A \subseteq U$ and $(V \cap A) \cap U \subseteq U \cap A$.

Theorem 2.4. A set $A \subseteq X$ is \mathcal{M} -collectionwise scattered in X if and only if $A = \bigcup N$ where N is a well ordered subcollection of \mathcal{M} , and for each $M \in N$ there exists an open set U_M such that $M \subseteq U_M$ and for all $K \in N$, if K does not precede M in N , then $K \cap U_M \subseteq M$.

Proof. Assume A is \mathcal{M} -collectionwise scattered, $A \neq \emptyset$, and $A = \bigcup N$ as in Definition 2.1. Let N be well ordered by the cardinal $|N| = \lambda$. Let M_0 denote the first element of N such that for some open $U \supseteq M_0$, $K \cap U \subseteq M_0$ for all $K \in N$. Let U_0 denote such an open set U . Assume M_β and U_β have been defined for all $\beta < \alpha$ where $\alpha < \lambda$. Then by Definition 2.1 we may define M_α as the first element of N such that for some open $U \supseteq M_\alpha$, if $K \in N \setminus \{M_\beta: \beta < \alpha\}$ then $K \cap U \subseteq M_\alpha$. Let U_α be such an open set. Thus sequences $\{M_\alpha: \alpha < \lambda\}$ and

$\{U_\alpha: \alpha < \lambda\}$ have been defined. If $M \in \mathcal{N}$, then for some $\alpha < \lambda$, $M = M_\alpha$. Thus $\mathcal{N} = \{M_\alpha: \alpha < \lambda\}$. Thus \mathcal{N} is expressed as a well ordered set. If $M_\alpha \in \mathcal{N}$, then for $\beta > \alpha$, $M_\beta \cap U_\alpha \subseteq M_\alpha$ by the construction.

Suppose the condition is satisfied by a well ordered collection \mathcal{N} . If $K \subseteq \mathcal{N}$ and $K \neq \emptyset$, let M be the first element of K . If $K \in K$ then K does not precede M . Hence $M \subseteq U_M$ and $K \cap U_M \subseteq M$.

Definition 2.5. A subset A of a space X is σ - \mathcal{M} -collectionwise scattered in X if and only if $A = \bigcup \{S_n: n \in \mathbb{N}\}$ where each S_n is \mathcal{M} -collectionwise scattered. If, in addition, each S_n is closed in X , then A is called σ -(closed and \mathcal{M} -collectionwise scattered).

Terminology 2.6. If $A \subseteq X$ is \mathcal{M} -collectionwise scattered then by Definition 2.1 and Theorem 2.4 there exist a collection \mathcal{N} and a collection $\mathcal{U} = \{U_M: M \in \mathcal{N}\}$ of open sets satisfying the conditions of Definition 2.1 or Theorem 2.4, respectively. A pair $\langle \mathcal{N}, \mathcal{U} \rangle$ of such collections \mathcal{N} and \mathcal{U} will be said to be an \mathcal{M} -pair for A , respectively, an \mathcal{M} -pair in the order sense for A , if the conditions of 2.1, respectively 2.4, are satisfied by \mathcal{N} and \mathcal{U} .

Theorem 2.7. Suppose X is a space and A is σ - \mathcal{R} -collectionwise scattered in X where \mathcal{R} is a disjoint collection of σ - \mathcal{M} -collectionwise scattered subsets of X . Then A is σ - \mathcal{M} -collectionwise scattered in X .

Proof. Suppose first that A is \mathcal{R} -collectionwise scattered. Let $\langle \mathcal{N}, \mathcal{U}(\mathcal{N}) \rangle$ be an \mathcal{R} -pair for A (see 2.6). For

each $M \in \mathcal{N}$, $M = \bigcup \{S(M, n) : n \in \omega\}$ and for each $n \in \omega$, $S(M, n)$ has an \mathcal{M} -pair $\langle \mathcal{M}'(M, n), \mathcal{U}(M, n) \rangle$. Let $K_n = \{K : \text{for some } M \in \mathcal{N}, K \in \mathcal{M}'(M, n)\}$. Suppose $\mathcal{L} \subseteq K_n$ and $\mathcal{L} \neq \emptyset$. Then $\mathcal{A} = \{M \in \mathcal{N} : \text{for some } K \in \mathcal{L}, K \in \mathcal{M}'(M, n)\} \neq \emptyset$. Hence there exists $A \in \mathcal{A}$ and $U_A \supseteq A$ such that U_A is open and $L \cap U_A \subseteq A$ for all $L \in \mathcal{A}$. Since \mathcal{R} is a disjoint collection, $L \cap U_A = \emptyset$ if $L \neq A$. Let $\mathcal{W} = \{K \in \mathcal{M}'(A, n) : K \in \mathcal{L}\}$. Then $\mathcal{W} \neq \emptyset$, so there exist $B \in \mathcal{W}$ and an open set $V_B \in \mathcal{U}(A, n)$ such that $B \subseteq V_B$ and $K \cap V_B \subseteq B$ for all $K \in \mathcal{W}$. Suppose $K \in \mathcal{L}$. If $K \in \mathcal{M}'(A, n)$, $K \cap V_B \cap U_A \subseteq B \cap U_A = B$, since $B \subseteq A$. If $K \in \mathcal{M}'(L, n)$ for some $L \in \mathcal{A}$ such that $L \neq A$, then $K \cap V_B \cap U_A \subseteq V_B \cap L \cap U_A = \emptyset \subseteq B$. Hence for each nonempty $\mathcal{L} \subseteq K_n$ there exist $B \in \mathcal{L}$ and an open set $W_B \supseteq B$ such that $L \cap W_B \subseteq B$ for all $L \in \mathcal{L}$. Hence $T_n = \bigcup K_n$ is \mathcal{M} -collectionwise scattered. Then $A = \bigcup \{T_n : n \in \omega\}$ is $\sigma\mathcal{M}$ -collectionwise scattered. If A is $\sigma\mathcal{R}$ -collectionwise scattered, the preceding argument shows that A is the countable union of $\sigma\mathcal{M}$ -collectionwise scattered sets.

Theorem 2.8. If X is a space and $A \subseteq X$ is \mathcal{R} -collectionwise scattered in X and \mathcal{R} is a disjoint collection of \mathcal{M} -collectionwise scattered subsets of X , then A is \mathcal{M} -collectionwise scattered.

Proof. Similar to the preceding theorem.

Theorem 2.9. Suppose X is a space, A is \mathcal{M} -collectionwise scattered in X ($\sigma\mathcal{M}$ -collectionwise scattered in X) and $B \subseteq X$ is such that $B \cap M \in \mathcal{M}$ for all $M \in \mathcal{M}$. Then $B \cap A$ is \mathcal{M} -collectionwise scattered in X ($\sigma\mathcal{M}$ -collectionwise scattered in X).

Proof. Suppose $\langle \mathcal{N}, \mathcal{U} \rangle$ is an \mathcal{M} -pair for A . Let

$\mathcal{N}|B = \{M \cap B: M \in \mathcal{N}\}$. Then $B \cap A = \bigcup \mathcal{N}|B$. If $K \neq \emptyset$ and $K \subseteq \mathcal{N}|B$, let $K' = \{M \in \mathcal{N}: M \cap B \in K\}$. Then $K' \neq \emptyset$ so there exists $M \in K'$ and an open $U_M \supseteq M$ such that for all $K \in K'$, $K \cap U_M \subseteq M$. Then $B \cap M \in K$ and if $B \cap K \in K$ where $K \in \mathcal{N}$, then $K \in K'$, so $B \cap K \cap U_M \subseteq B \cap M$. Thus the first statement is proved. If A is a countable union of \mathcal{M} -collectionwise scattered sets S_n , then each $B \cap S_n$ is \mathcal{M} -collectionwise scattered.

Theorem 2.10. Suppose X is a space and $A \subseteq X$ is covered by a collection \mathcal{V} of open sets such that for all $V \in \mathcal{V}$, $V \cap A$ is $\sigma\mathcal{M}$ -collectionwise scattered (\mathcal{M} -collectionwise scattered). Suppose that for every closed $F \subseteq X$, $F \cap M \in \mathcal{M}$ for all $M \in \mathcal{M}$. Then A is $\sigma\mathcal{M}$ -collectionwise scattered (\mathcal{M} -collectionwise scattered).

Proof. Well order \mathcal{V} and let $\mathcal{V}' = \{V \in \mathcal{V}: p(V, \mathcal{V}) \cap A \neq \emptyset\}$ where $p(V, \mathcal{V}) = V \cup \{V': V' \text{ precedes } V \text{ in } \mathcal{V}\}$. Let $\mathcal{N} = \{p(V, \mathcal{V}) \cap A: V \in \mathcal{V}' \setminus \{\emptyset\}\}$. Then \mathcal{N} is a disjoint collection of sets which are $(\sigma)\mathcal{M}$ -collectionwise scattered by the preceding theorem. Theorem 2.7 (2.8) implies that A is $\sigma\mathcal{M}$ -collectionwise scattered (\mathcal{M} -collectionwise scattered).

Corollary 2.11. If X is locally \mathcal{M} -collectionwise scattered ($\sigma\mathcal{M}$ -collectionwise scattered) where \mathcal{M} is relatively closed hereditary (cf. Theorem 2.10), then X is \mathcal{M} -collectionwise scattered ($\sigma\mathcal{M}$ -collectionwise scattered).

If X has property P locally and P is a closed hereditary property, then X is collectionwise \mathcal{M} -scattered, where \mathcal{M} is a disjoint collection of sets each of which has property P .

3. Primitive Sequences

We recall here some primitive sequence theory [WW₂, WW₃, WW₆] in order to facilitate the proofs of the main results.

Definition 3.1. Let (Z, \leq) be a well ordered collection of sets. For each $W \in A$, let $p(w, A)$ denote $W \setminus \{W' \in Z: W' < W\}$. The set $p(w, Z)$ is called the *primitive part* of W (in Z).

For $x \in \cup Z$, we let $F(x, Z)$ denote the first element of Z that contains x . $F(A, Z)$ is similarly defined for $A \subseteq W \in Z$.

Definition 3.2. A *primitive sequence* \mathcal{H} of A in X is a sequence $\mathcal{H} = \langle H_n: n \in \mathbb{N} \rangle$ of well ordered subcollections of $\mathcal{P}(X)$ such that for all $n \in \mathbb{N}$:

- (1) For all $H \in H_n$, $A \cap p(H, H_n) \neq \emptyset$.
- (2) For all $x \in A$, $F(x, H_{n+1}) \subseteq F(x, H_n)$.

An *open primitive sequence* is a primitive sequence relative to a topological space X whose terms are well ordered collections of sets open in X .

Definition 3.3. Let \mathcal{H} be a primitive sequence of A in X . A *primitive representative* of \mathcal{H} is a sequence $H = \langle H_n: n \in \mathbb{N} \rangle$ such that for all $n \in \mathbb{N}$, $p(H_n, H_n) \cap p(H_{n+1}, H_{n+1}) \neq \emptyset$. The collection of all primitive representatives of \mathcal{H} will be denoted by $PR(\mathcal{H})$.

If $\mathcal{G} = \langle G_n: n \in \mathbb{N} \rangle$ is a sequence of collections of sets a *decreasing representative* of \mathcal{G} is a sequence G such that $G_{n+1} \subseteq G_n \in \mathcal{G}_n$ for all $n \in \mathbb{N}$. The set of all decreasing representatives of \mathcal{G} will be denoted by $DR(\mathcal{G})$.

If $H = \langle H_n: n \in \mathbb{N} \rangle$ and $G = \langle G_n: n \in \mathbb{N} \rangle$ are decreasing

sequences, H is said to *dominate* G (or G *refines* H) if and only if for all $n \in \mathbb{N}$ there is $j > n$ such that $G_j \subseteq H_n$.

4. Scattering and Primitive Sequences

We present theorems here guaranteeing the existence of primitive sequences for sets which are scattered with respect to collections of sets having primitive sequences. These theorems underlie the proofs of the results of section 5.

Theorem 4.1. Suppose X is a space, \mathcal{A} is \mathcal{M} -collectionwise scattered in X and $\langle \mathcal{N}, \mathcal{U} \rangle$ is an \mathcal{M} -pair for \mathcal{A} in the order sense such that each $M \in \mathcal{N}$ has an open primitive sequence $V(M)$ in U_M . Then there exists an open primitive sequence \mathcal{H} of \mathcal{A} in X such that for all $H \in \text{DR}(\mathcal{H})$, there exist $k \in \mathbb{N}$ and $M \in \mathcal{N}$ such that $H_j = V_j$ for all $j \geq k$ for some $V \in \text{DR}(V(M))$.

Proof. Using the notation in the statement, let $\mathcal{H}_n = \{H: \text{for some } M \in \mathcal{N}, H \in V(M)_n \text{ and } p(M, \mathcal{N}) \cap p(H, V(M)_n) \neq \emptyset\}$. If $H \in \mathcal{H}_n$ there is only one M associated with H as indicated. For if $p(M, \mathcal{N}) \cap p(H, V(M)_n) \neq \emptyset$, and $p(M', \mathcal{N}) \cap p(H, V(M')_n) \neq \emptyset$ and $H \in V(M)_n$, then $H \subseteq U_M$. Hence $H \cap M' \subseteq M$ by Theorem 2.4. Since $H \cap p(M', \mathcal{N}) \neq \emptyset$, it follows that M' does not follow M . Similarly, M does not follow M' so $M = M'$. Temporarily denote such an M by $M(H)$. For H and H' in \mathcal{H}_n define $H <_n H'$ if and only if $M(H)$ precedes $M(H')$ in \mathcal{N} or $M(H) = M(H')$ and H precedes H' in $V(M(H))_n$. It may be readily verified that \leq_n well orders \mathcal{H}_n and \mathcal{H}_n covers \mathcal{A} . Moreover, if $x \in \mathcal{A}$ and $M = F(x, \mathcal{N})$, then $F(x, \mathcal{H}_n) = F(x, V(M)_n)$. From this it is easy to verify that $\mathcal{H} = \langle \mathcal{H}_n: n \in \mathbb{N} \rangle$ is an open primitive sequence of \mathcal{A} in X . Suppose $H \in \text{DR}(\mathcal{H})$. Then for each $n \in \mathbb{N}$, $H_n \in V(M_n)_n$, where $M_n = M(H_n)$. In the well ordering of \mathcal{N} , let M_k

be the first element of the set of M_n 's. If $j > k$, there exists $x_j \in p(H_j, \mathcal{V}(M_j)_j) \cap p(M_j, \mathcal{M})$. Since $x_j \in H_j \subseteq H_k$, $x_j \in M_j \cap U_{M_k} \subseteq M_k$. Thus $M_k = M_j$. Hence $H_j \in \mathcal{V}(M_k)_j$ for all $j \geq k$. Let $V_j = H_j$ for $j \geq k$ and define V_j inductively for $j < k$, by $V_j = F(V_{j+1}, \mathcal{V}(M_k)_j)$. Then $V \in DR(\mathcal{V}(M_k))$.

Remark 4.2. If A is \mathcal{M} -collectionwise scattered in X and each $M \in \mathcal{M}$ has an open primitive sequence $\mathcal{V}(M)$ in X , then there exists an open primitive sequence \mathcal{H} of A in X and for each $M \in \mathcal{M}$ an open set $U_M \supseteq M$ such that for all $H \in DR(\mathcal{H})$, there exist $V \in DR(\mathcal{V}(M)|_{U_M})$ and $k \in \mathbb{N}$ such that $V_j = H_j$ for all $j \geq k$.

Theorem 4.3. Suppose X is a space, A is \mathcal{M} -collectionwise scattered in X and each $M \in \mathcal{M}$ has an open primitive sequence $\mathcal{V}(M)$ in X . Then A has an open primitive sequence \mathcal{H} in X such that for all $H \in DR(\mathcal{H})$, there exist M and $V \in DR(\mathcal{V}(M))$ such that V dominates H .

Proof. Use the notation of Theorem 4.1, with the exception that $\mathcal{V}(M)$ need no longer be in U_M . Define $K(M)_n = \mathcal{V}(M)_n|_{U_M} = \{V \cap U_M : V \in \mathcal{V}(M)_n\}$ and note that $K \mapsto$ first element of $\{V \in \mathcal{V}(M)_n : V \cap U_M = K\}$ is an injection of $K(M)_n$ into $\mathcal{V}(M)_n$. Using the order on $K(M)_n$ induced by this mapping, it is easily seen that $\langle K(M)_n : n \in \mathbb{N} \rangle$ is a primitive sequence of M in U_M . Now Theorem 4.1 applies and we obtain an open primitive sequence \mathcal{H} relative to the $K(M)$ sequences. If $H \in DR(\mathcal{H})$ there is M and $K \in DR(K(M))$ with the property indicated in Theorem 4.1. Hence there exists $L \in PR(K(M))$ that dominates K and hence H (take $L_n =$ first element of $K(M)_n$ that includes a term of K). It is also easy to see that

there is $V \in PR(\mathcal{V}(M))$ that dominates L . Hence the result.

The following simple lemmas are used to abbreviate the proof of the next theorem.

Lemma 4.4. Suppose Z_1, \dots, Z_k are well ordered collections of sets. Let $r_k(M_1, \dots, M_k)$ denote the relation $M_i \in Z_i$ for $i = 1, \dots, k$ and $\cap\{p(M_i, Z_i) : i \leq k\} \neq \emptyset$. Then if $M_1 \cap \dots \cap M_k = M'_1 \cap \dots \cap M'_k$ where $M_i, M'_i \in Z_i$ for all $i \leq k$ and $r_k(M_1, \dots, M_k)$ and $r_k(M'_1, \dots, M'_k)$ hold, then $M_i = M'_i$ for all $i \leq k$.

Proof. Since $\cap\{p(M_i, Z_i) : i \leq k\} \neq \emptyset$ and $\cap\{p(M'_i, Z_i) : i \leq k\} \neq \emptyset$, it follows that for each $i \leq k$, $M_i \cap p(M'_i, Z_i) \neq \emptyset \neq M'_i \cap p(M_i, Z_i)$. Hence $M_i = M'_i$ for all $i \leq k$.

Lemma 4.5. Suppose Z_1, \dots, Z_k are well ordered collections of sets. Then the set $\mathcal{W} = \{M_1 \cap \dots \cap M_k : r_k(M_1, \dots, M_k)\}$ has a well ordering determined by the orderings on the Z_i such that for all $x \in \mathcal{W}$,

$$F(x, \mathcal{W}) = F(x, Z_1) \cap \dots \cap F(x, Z_k).$$

Proof. By the preceding lemma, the expression $M_1 \cap \dots \cap M_k$ is unique for elements of \mathcal{W} . For $W, W' \in \mathcal{W}$, define $W < W'$ if and only if $W \neq W'$ and if i is the first j such that $M_j \neq M'_j$, then M_i precedes M'_i in Z_i . The verification that \leq well orders \mathcal{W} is straightforward. Suppose $x \in \mathcal{W}$ and $F(x, \mathcal{W}) = M_1 \cap \dots \cap M_k$. Since $F(x, Z_i) \subseteq M_i$ (in Z_i), it follows that $M_i = F(x, Z_i)$.

Remark 4.6. We will refer to well orderings given by Lemma 4.5 as "natural well orderings." Note that in the application of Lemma 4.5 in the proof below some of the sets

Z_i consist of a single element; we consider such a Z_i as a well ordered set of one element. We will well order finite unions of disjoint well ordered sets below in the standard way: If $W = U_1 \cup \dots \cup U_n$, the elements of U_i precede those of U_j for $i < j$ and the elements of each U_i retain the order of U_i .

Theorem 4.7. Suppose X is a space, $A \subseteq X$, and $A = \bigcup \{S_n : n \in \mathbb{N}\}$ where each S_n is closed and has an open primitive sequence $W^n = \langle W_m^n : m \in \mathbb{N} \rangle$ in X . Then A has an open primitive sequence \mathcal{H} in X such that if $G \in \text{DR}(\mathcal{H})$ and $\bigcap \{G_n : n \in \mathbb{N}\} \neq \emptyset$, then there exist $k \in \mathbb{N}$ and $W \in \text{PR}(W^k)$ such that W dominates G .

Proof. Define $F_0 = \emptyset$ and $F_n = S_1 \cup \dots \cup S_n$. Define $V_m^k = \{W \setminus F_{k-1} : W \in W_m^k \text{ and } p(W, W_m^k) \cap (S_k \setminus F_{k-1}) \neq \emptyset\}$. Lemma 4.5 applies, so we give V_m^k the natural well ordering. If V_1^1 covers A , define \mathcal{J}_1 as \emptyset , otherwise let $\mathcal{J}_1 = \{X \setminus S_1\}$. Define $\mathcal{H}_1 = V_1^1 \cup \mathcal{J}_1$ and well order it as a well ordered union. Then \mathcal{H}_1 is critically well ordered (i.e. well ordered and $p(H, \mathcal{H}_1) \neq \emptyset$ for all $H \in \mathcal{H}_1$) and covers A . Assume $\mathcal{H}_1, \dots, \mathcal{H}_{n-1}$ have been defined and are critically well ordered open covers of A .

For $k \leq n$ and $m \leq n$ and $k + m = n + 1$ define $\beta_m^k = \{V \cap H : V \in V_m^k, H \in \mathcal{H}_{n-1}, \text{ and } p(V, V_m^k) \cap p(H, \mathcal{H}_{n-1}) \cap (S_k \setminus F_{k-1}) \neq \emptyset\}$. Give each β_m^k the natural well ordering and let $\mathcal{H}'_n = \beta_1^n \cup \beta_2^{n-1} \cup \dots \cup \beta_n^1$ well ordered as a disjoint union. Let $\mathcal{J}_n = \{H \setminus F_n : H \in \mathcal{H}_{n-1} \text{ and } p(H, \mathcal{H}_{n-1}) \cap A \setminus F_n \not\subseteq \bigcup \mathcal{H}'_n\}$ with the natural well ordering.

Let $\mathcal{H}_n = \mathcal{H}'_n \cup \mathcal{J}_n$ well ordered as a disjoint union. Then

\mathcal{H}_n is a critically well ordered open cover of A and it may be verified that $\mathcal{H} = \langle \mathcal{H}_n : n \in \mathbb{N} \rangle$ is an open primitive sequence of A in X .

Suppose $G \in DR(\mathcal{H})$ and $\cap \{G_n : n \in \mathbb{N}\} \neq \emptyset$. Let k be the first integer j such that for some $x \in \cap \{G_n : n \in \mathbb{N}\}$, $x \in S_j$. Then $x \in S_k \setminus F_{k-1}$ and $x \notin \bigcup_m V_m^i$ for any $i > k$. Moreover $x \notin \bigcup_n J_n$ for $n > k$. Thus for $n > k$ there exists $j \leq k$ such that $G_n \in \beta_{n-j+1}^j$. Since $G_n \in \beta_m^j$ implies that $G_{n+1} \notin \beta_m^{j'}$ for $j' < j$, it follows that for some $m \geq k$ there exists $t \leq k$ such that $G_n \in \beta_{n-t+1}^t$ for all $n \geq m$. From this it follows that for each $n \in \mathbb{N}$ there exists W_n such that W_n is the first element of \mathcal{W}_n^t that includes some term of G . By a standard argument (Lemma 2.2 of [WW₂]) it follows that $W \in PR(\mathcal{W}^t)$.

Theorem 4.8. Let X be a space, and suppose that A is σ -(closed and \mathcal{M} -collectionwise scattered) in X and each $M \in \mathcal{M}$ has an open primitive sequence $\mathcal{V}(M)$ in X . Then there exists an open primitive sequence \mathcal{H} of A in X such that for all $H \in DR(\mathcal{H})$, if $\cap \{H_n : n \in \mathbb{N}\} \neq \emptyset$, then there exist M and $V \in PR(\mathcal{V}(M))$ such that V dominates H .

Proof. The set $A = \bigcup \{S_n : n \in \mathbb{N}\}$, where each S_n is closed and \mathcal{M} -collectionwise scattered. By Theorem 4.3, each S_n has an open primitive sequence \mathcal{W}^n such that for each $W \in DR(\mathcal{W}^n)$ there is M and $V \in PR(\mathcal{V}(M))$ such that V dominates W . By Theorem 4.7, A has an open primitive sequence \mathcal{H} in X such that for all $H \in DR(\mathcal{H})$ if $\cap \{H_n : n \in \mathbb{N}\} \neq \emptyset$, then there exist n and $W \in PR(\mathcal{W}^n)$ such that W dominates H . Thus there exist M and $V \in PR(\mathcal{V}(M))$ such that V dominates W and, hence V dominates H .

Theorem 4.9. *Let X be a space and suppose that A is $\sigma\mathcal{M}$ -collectionwise scattered in X and each $M \in \mathcal{M}$ has an open primitive sequence $V(M)$ in X . Then there exists an open primitive sequence \mathcal{H} of A in X such that for all $H \in \text{PR}(\mathcal{H})$, if $\text{pc}(H) = \cap \{p(H_n, \mathcal{H}_n) : n \in \mathbb{N}\} \neq \emptyset$, then there exist M and $V \in \text{PR}(V(M))$ such that V dominates H and $\text{pc}(H) \subseteq \text{pc}(V)$.*

Proof. The proof is similar to, but simpler than that of Theorem 4.7. A is the union $\cup \{S_n : n \in \mathbb{N}\}$ of \mathcal{M} -collectionwise scattered sets. Each S_n has an open primitive sequence \mathcal{W}^n formed as in Theorem 4.3. Let $\mathcal{H}_1 = \mathcal{W}_1^1$ with X added as a final element if \mathcal{W}_1^1 does not cover A . Assuming $\mathcal{H}_1, \dots, \mathcal{H}_{n-1}$ to be critically well ordered sets covering A , for $k \leq n$ and $m \leq n$ and $k + m = n + 1$, define $\beta_m^k = \{V \cap H : V \in \mathcal{W}_m^k, H \in \mathcal{H}_{n-1}\}$, and $p(V, \mathcal{W}_m^k) \cap p(H_{n-1}) \cap (S_k \setminus F_{k-1}) \neq \emptyset$, where $F_0 = \emptyset$ and $F_n = S_1 \cup \dots \cup S_n$. Let $\mathcal{H}_n = \beta_1^n \cup \beta_2^{n-1} \cup \dots \cup \beta_n^1$ with the natural well ordering (together with X as a final element if \mathcal{H}_n does not cover A). It may be verified that \mathcal{H} is an open primitive sequence which satisfies the condition.

5. Base of Countable Order Theory

We briefly summarize some concepts used in the theorems below, and prove three further theorems useful in the proofs.

Suppose H is a sequence. If $\{H_n : n \in \mathbb{N}\}$ consists of open sets which are a base at a point x of a space, we say H is a base at x .

We list the following properties of decreasing sequences $B = \langle B_n : n \in \mathbb{N} \rangle$ of open sets in a space X and $A \subseteq X$ (cf. [CCN] and $[W_4]$):

Definition 5.1.

- (λ) There exists $x \in X$ such that every open set containing x includes some B_n and B is a base at all elements of $\cap\{B_n: n \in \mathbb{N}\}$.
- (b) B is a base at all elements of $\cap\{B_n: n \in \mathbb{N}\}$.
- (p) If \mathcal{F} is a filterbase in X and \mathcal{F} refines B (i.e. each B_n includes some $F \in \mathcal{F}$), then $\cap\{\bar{F}: F \in \mathcal{F}\} \neq \emptyset$.
- (q) If \mathcal{F} is a countable filterbase in X and \mathcal{F} refines B , then $\cap\{\bar{F}: F \in \mathcal{F}\} \neq \emptyset$.
- (d) $\cap\{B_n: n \in \mathbb{N}\}$ is empty or a singleton.
- (sA) $\cap\{B_n: n \in \mathbb{N}\} \subseteq A$.

Remark 5.2. Note that all properties listed are *monotonic* [CCN]: A property (M) of decreasing sequences B is *monotonic* if and only if for every decreasing sequence W of open sets, if B dominates W and B has (M), then W has (M). This fact, together with the nature of the definitions below, permits the theorems below to be derived from those of section 4.

Now we define certain types of open primitive sequences. Suppose \mathcal{W} is an open primitive sequence of A in X . Let

$$PR(\mathcal{W}, A) = \{W \in PR(\mathcal{W}): p(W_n, \mathcal{W}_n) \cap A \neq \emptyset \text{ for all } n \in \mathbb{N}\},$$

$$PR^\mu(\mathcal{W}, A) = \{W \in PR(\mathcal{W}, A): A \cap \cap\{W_n: n \in \mathbb{N}\} \neq \emptyset\},$$

$$PR^\pi(\mathcal{W}, A) = \{W \in PR(\mathcal{W}, A): A \cap \cap\{p(W_n, \mathcal{W}_n): n \in \mathbb{N}\} \neq \emptyset\}.$$

Definition 5.3. Let (c) denote any of the properties listed in Definition 5.1. Then an open primitive sequence \mathcal{W} of A in X is called:

a (c)-sequence of A in X if and only if every $W \in PR(W, A)$ has (c),

a (μ c)-sequence of A in X if and only if every $W \in PR^\mu(W, A)$ has (c),

a (π c)-sequence of A in X if and only if every $W \in PR^\pi(W, A)$ has (c).

If $A = X$ we speak of a (c)-sequence in X , etc.

Remark 5.4. Note that if M has an (α) -sequence in X , where (α) is any of the properties listed in the preceding definition, then if $F \subseteq X$ is closed, $F \cap M$ has an (α) -sequence in X .

The following theorem summarizes theorems and definitions of $[WW_3, WW_2]$ and the present paper.

Theorem 5.5. An essentially T_1 space has the property listed below on the left if and only if it has an open primitive (α) -sequence where (α) is listed at right.

λ -base	(λ)
base of countable order	(μ b)
primitive base	(π b)
primitively quasi-complete (or primitively q-space)	(π q)
primitively p-space	(π p)
primitive diagonal	(π d)
diagonal a set of interior condensation	(γ d).
If X is pararegular $[WW_6]$,	
β_b -space	(μ p)
β_c -space	(μ q)
λ_b -space	(p)

λ_C -space

(q)

If $A \subseteq X$ then A has the property

set of interior condensation

(μSA).

We add the following definitions as natural accompaniments to the foregoing (see Remark 5.9).

Definition 5.6. A space X is said to be:

a *monotonically p-space* if and only if it has a (μp) -sequence;

a *monotonically q-space* if and only if it has a (μq) -sequence.

A space X is said to be:

(p)-*complete* if and only if it has a (p)-sequence;

(q)-*complete* if and only if it has a (q)-sequence.

The above concepts coincide, respectively, with $\beta_b, \beta_C, \lambda_b, \lambda_C$ -spaces $[W_1]$, when X is pararegular $[WW_6]$.

The following definition is useful in stating the main results.

Definition 5.7. Let X be a space and $M \subseteq X$. Let P be any of the properties listed in Theorem 5.5 or Definition 5.6 defined by some property (α) . We say that M has P in the *global topology* if and only if there is an (α) -sequence of M in X .

Remarks 5.8.

Thus for example, M has a base of countable order in the global topology if and only if there exists a sequence $\mathcal{G} = \langle \mathcal{G}_n : n \in \mathbb{N} \rangle$ of open sets such that each \mathcal{G}_n is a basis for the topology of X at all points of M and if G is a decreasing representative of \mathcal{G} such that $x \in \bigcap \{G_n : n \in \mathbb{N}\} \cap M$,

then $\{G_n: n \in \mathbb{N}\}$ is a base at x .

If X is a first countable space then each $\{x\} \subseteq X$ has a base of countable order (in fact a strong λ -base (see 6)) in the global topology. If X is of point countable type (respectively, a q -space), then each $\{x\} \subseteq X$ is (p) -complete, $((q)$ -complete) in the global topology.

We note that the concepts of primitive base, primitively quasi-complete, and primitively p -space can be formulated more simply in terms of a sequence \mathcal{W} of well ordered open covers of X by requiring an appropriate condition on $\{F(x, \mathcal{W}_n): n \in \mathbb{N}\}$ for all $x \in X$.

Thus X has a primitive base if and only if there exists a sequence as described such that for all $x \in X$, $\{F(x, \mathcal{W}_n): n \in \mathbb{N}\}$ is a base at x .

Remarks 5.9.

The following remarks are intended to help in organizing certain of the previously named concepts. Consider the four properties: (1) space of point countable type $[A_1]$, (2) q -space $[M]$, (3) first countable space, (4) points are G_δ 's. The monotonic versions of these properties are, respectively: (M1) monotonically p -space, (M2) monotonically q -space, (M3) spaces having bases of countable order, (M4) diagonal a set of interior condensation. The primitive versions are, respectively: (p1) primitively p -space, (p2) primitively q -space, (p3) primitive base, (p4) primitive diagonal. Thus some of the most frequently used monotone and primitive spaces are uniformizations of first countability or some of its generalizations.

In the following theorems let X be a space, $A \subseteq X$, and \mathcal{M} a collection of subsets of X .

Theorem 5.10. Suppose A is \mathcal{M} -collectionwise scattered in X and each $M \in \mathcal{M}$ has an (α) -sequence in X where (α) is any of the properties of the right hand list in Theorem 5.5. Then each $M \in \mathcal{M}$ has an (α) -sequence $V(M)$ and A has an (α) -sequence \mathcal{H} such that for all $H \in \text{DR}(\mathcal{H})$ there is some $M \in \mathcal{M}$ and $k \in \mathbb{N}$ such that $H_j = V_j$ for all $j \geq k$ and some $V \in \text{DR}(V(M))$.

Proof. If $\langle \mathcal{N}, \mathcal{U} \rangle$ is an \mathcal{M} -pair for A in the order sense, we can, using the technique of proof of Theorem 4.7, obtain an (α) -sequence $V(M)$ of M in U_M for all $M \in \mathcal{N}$. By Theorem 4.1, there is an open primitive sequence \mathcal{H} of A in X such that each $H \in \text{DR}(\mathcal{H})$ satisfies the stated condition. Since each $V(M)$ is an (α) -sequence, each $H \in \text{DR}(\mathcal{H})$ will have (α) , because the $V \in \text{DR}(V(M))$ given by the theorem has (α) .

Theorem 5.11. Suppose A is σ -(closed and \mathcal{M} -collectionwise scattered) in X and each $M \in \mathcal{M}$ has an (α) -sequence in X , where (α) is any of the properties of Theorem 5.5 involving μ or π . Then A has an (α) -sequence in X .

Proof. Let $V(M)$ be an (α) -sequence for each $M \in \mathcal{M}$. There exists an open primitive sequence \mathcal{H} of A in X satisfying the conclusion of Theorem 4.8. Suppose $H \in \text{DR}(\mathcal{H})$ and $\bigcap \{H_n : n \in \mathbb{N}\} \neq \emptyset$. Then if $V \in \text{PR}(V(M))$ dominates H , $V \in \text{PR}(\mathcal{H})$. Thus if $(\alpha) = (\mu c)$ or (πc) , V has (c) and thus by Remark 5.2, so does H .

Theorem 5.12. If A is σ - \mathcal{M} -collectionwise scattered and

each $M \in \mathcal{M}$ has an (α) -sequence where (α) is a property listed in 5.3 involving π , then A has an (α) -sequence.

Proof. Each $M \in \mathcal{M}$ has an (α) -sequence $V(M)$ in X . Hence there is an open primitive sequence \mathcal{H} of A in X satisfying the condition of Theorem 4.9. If $H \in \text{PR}^\pi(\mathcal{H})$, then there is M and $V \in \text{PR}(V(M))$ such that V dominates H . Since $\text{pc}(V) \neq \emptyset$, it follows that if $V(M)$ has (πc) then V has (c) and so does H . Hence \mathcal{H} has (πc) .

6. Main Results

In the theorems below, we assume that the set \mathcal{M} is a relatively closed hereditary subcollection of the power set of a space X . This is justified in view of Remark 5.4. See 5.7 for explanation of the usage of *global topology*.

We say that X has a *strong λ -base* if and only if X has a (λ) -sequence \mathcal{H} such that if $H \in \text{PR}(\mathcal{H})$, then $\bigcap \{H_n : n \in \mathbb{N}\} \neq \emptyset$. A set $M \subseteq X$ has a *strong λ -base in the global topology* provided it has a (λ) -sequence \mathcal{H} in X such that for all $H \in \text{PR}(\mathcal{H})$, $M \cap \bigcap \{H_n : n \in \mathbb{N}\} \neq \emptyset$.

Theorem 6.1. Suppose X is a locally \mathcal{M} -collectionwise scattered space.

(a) If each $M \in \mathcal{M}$ has a strong λ -base (λ -base) in the global topology, then X has a strong λ -base (λ -base).

(b) If each $M \in \mathcal{M}$ is (p) -complete ((q) -complete) in the global topology, then X is (p) -complete ((q) -complete). If X is pararegular and each $M \in \mathcal{M}$ is a $\lambda_b - (\lambda_c)$ -space, then X is a $\lambda_b - (\lambda_c)$ -space.

Proof. (a) By Corollary 2.1 and our blanket assumption on \mathcal{M} , X is \mathcal{M} -collectionwise scattered. Each $M \in \mathcal{M}$ has a

(λ) -sequence (respectively, a strong λ -sequence). By Theorem 5.10, X has the same kind of sequence. The proof of part (b) is similar.

Corollary 6.2. Suppose X is a locally scattered T_1 -space. If X is first countable, then X has a strong λ -base. If X is T_2 of point countable type, then X is (p) -complete. If X is a q -space, then X is (q) -complete.

Proof. Here $\mathcal{M} = \{\{x\} : x \in X\}$. By Remarks 5.8 the hypotheses of the theorem are satisfied.

Theorem 6.3. Suppose X is locally σ -(closed in X and \mathcal{M} -collectionwise scattered).

(a) If each $M \in \mathcal{M}$ has a base of countable order in the global topology, then X has a base of countable order.

(b) If each $M \in \mathcal{M}$ is a monotonically p -space (respectively, a monotonically q -space) in the global topology, then X is a monotonically p -space (respectively, a monotonically q -space). If X is pararegular and each $M \in \mathcal{M}$ is a β_p -space (β_c -space) in the global topology, then X is a β_p -space (β_c -space).

(c) If each $M \in \mathcal{M}$ has diagonal a set of interior condensation, then X also has such a diagonal.

Proof. (a) Suppose $U \subseteq X$ is open and σ -(closed in X and \mathcal{M} -collectionwise scattered). Since each $M \in \mathcal{M}$ has a base of countable order in the global topology, each $M \in \mathcal{M}$ has a (μb) -sequence in X . By Theorem 5.11, U has a (μb) -sequence in X . The conclusion follows from the local-implies-global property of base of countable order [WoW]. It may also be deduced from Corollary 2.11 of the present paper.

The proofs of (b) and (c) are similar.

Corollary 6.4. Suppose X is a locally σ -(closed in X and scattered) T_1 space.

(a) If X is first countable, then X has a base of countable order.

(b) If X is T_2 and of point countable type (a q -space) then X is a monotonically p -space (q -space).

(c) If points are G_δ 's in X , then X has diagonal a set of interior condensation.

Proof. Here again take $\mathcal{M} = \{\{x\}: x \in X\}$. The conditions stated then translate into the corresponding conditions of the preceding theorem for the sets $M = \{x\} \in \mathcal{M}$.

Theorem 6.5. Suppose X is a space and $A \subseteq X$. If A is σ -(closed and \mathcal{M} -collectionwise scattered) in X where each $M \in \mathcal{M}$ is a set of interior condensation, then A is a set of interior condensation.

Proof. Each $M \in \mathcal{M}$ has a $(\mu s M)$ -sequence in X . By Theorem 5.11, A has a $(\mu s A)$ -sequence in X , so that A is a set of interior condensation, by 5.3.

Theorem 6.6. Suppose X is a locally σ - \mathcal{M} -collectionwise scattered space.

(a) If each $M \in \mathcal{M}$ has a primitive base in the global topology then X has a primitive base.

(b) If each $M \in \mathcal{M}$ is a primitively p -space (q -space) in the global topology, then X is a primitively p -space (q -space).

(c) If each $M \in \mathcal{M}$ has a primitive diagonal in the global topology, then X has a primitive diagonal.

Proof. By Corollary 2.11, X is σ - \mathcal{M} -collectionwise

scattered. Since each $M \in \mathcal{M}$ has an appropriate type of (α) -sequence in X as listed in Theorem 5.12, it follows that X has such an (α) -sequence. Application of Theorem 5.3 completes the proof.

Corollary 6.7. *Let X be a locally σ -scattered space.*

- (a) If X is first countable, then X has a primitive base.*
- (b) If X is T_2 and of point countable type (a q -space) then X is a primitively p -space (q -space).*
- (c) If points are G_δ 's in X , then X has a primitive diagonal.*

Proof. This follows from Theorem 6.6, analogously to the way in which Corollary 6.4 follows from Theorem 6.3.

7. Applications and Examples

We make application here to the case of scattered and σ -scattered spaces as illustrations.

Theorem 7.1. *Let X be a T_1 locally scattered q -space.*

- (a) If points are G_δ 's and X is hereditarily weakly θ -refinable then X is quasi-developable and has a strong λ -base.*
- (b) If closed sets are G_δ 's and X is weakly θ -refinable, then X is developable and semi-complete.*
- (c) If X is collectionwise normal, closed sets are G_δ 's, and X is weakly θ -refinable, then X is completely metrizable.*

Proof. Since X is locally scattered, it is scattered. Since X is a T_1 , q -space and thus in (a), (b), and (c) points

are G_δ 's, it follows that X is T_1 and first countable [L]. Hence by 6.2, X has a strong λ -base. Part (a) then follows from [BB]. Part (b) follows from (a) and [BL]. Part (c) follows from (b) and [B] and the fact that a metric space with a λ -base is completely metrizable [WW₁].

Theorem 7.2. Suppose X is a regular locally σ -(closed and scattered) space such that points are G_δ 's. Then X has a base of countable order if and only if X is a monotonically p -space.

Proof. By 6.4 (c), X has diagonal a set of interior condensation. A regular monotonically p -space with such a diagonal has a base of countable order [W₃].

Theorem 7.3. Let X be a locally σ -(closed and scattered) regular T_1 q -space. Then if X is θ -refinable, it is a p -space.

Proof. By 6.4 (b), X is a monotonically q -space. By [WW₅], it is a p -space.

Theorem 7.4. Let X be a scattered Tychonoff space of point countable type. If X is θ -refinably embedded in X , then X is Čech complete.

Proof. By 6.2, X is a λ_b -space. The result follows from [WW₅].

Theorem 7.5. Suppose X is a locally countable T_1 q -space. Then X has a base of countable order.

Proof. Since X is locally countable, it is locally σ -scattered, hence σ -scattered. Thus locally it has a primitive base and closed sets are G_δ 's. By [WW₄], locally it

has a base of countable order and thus has one globally [WoW].

Theorem 7.6. Suppose X is a locally σ -scattered T_1 q -space. If closed sets are sets of interior condensation in X , then X has a base of countable order.

Proof. By 6.7 such a space is T_1 and has a primitive base because the hypothesis implies first countability. By [WW₄], the space has a base of countable order.

That the classes of scattered, σ -scattered, σ -(closed and scattered) first countable spaces are distinct is shown by the following examples.

Example 7.7. The space Q of rationals with the usual topology is a σ -(closed and scattered) space which is not scattered. It does not have a λ -base since it doesn't have the Baire property.

Example 7.8. The so-called Michael line [SS, p. 90] is a σ -scattered first countable T_2 paracompact space which is not σ -(closed and scattered). This follows from the fact that it cannot have a base of countable order.

Example 7.9. The space ω_1 with the order topology shows that no stronger property such as developable or even quasi-developable is implied by being scattered, normal, and first countable.

References

- [A₁] A. V. Arhangel'skiĭ, *Bicompact sets and the topology of spaces*, Tr. Mosk. Mat. Obšč. 13 (1965), 3-55.
- [BB] H. R. Bennett and E. S. Berney, *On certain generalizations of developable spaces*, Gen. Top. Appl. 4 (1974), 43-50.

- [BL] H. R. Bennett and D. J. Lutzer, *A note on weak θ -refinability*, Gen. Top. Appl. 2 (1972), 49-54.
- [B] R. H. Bing, *Metrization of topological spaces*, Can. J. Math. 3 (1951), 175-186.
- [CCN] J. Chaber, M. M. Čoban and K. Nagami, *On monotonic generalizations of Moore spaces, Čech complete spaces, and p-spaces*, Fund. Math. 84 (1974), 107-119.
- [K] K. Kuratowski, *Topology I*, Academic Press, New York, 1966.
- [L] D. J. Lutzer, *Semimetrizable and stratifiable spaces*, Gen. Top. Appl. 1 (1971), 43-48.
- [M] E. Michael, *A note on closed maps and compact sets*, Israel J. Math. 2 (1964), 173-176.
- [SS] L. A. Steen and J. A. Seebach, Jr., *Counter-examples in topology*, Holt, Rinehart, and Winston, New York, 1970.
- [T] R. Telgársky, *Total paracompactness and paracompact dispersed spaces*, Bull. Acad. Pol. Sci. 16 (1968), 567-572.
- [W₁] H. H. Wicke, *Open continuous images of certain kinds of M-spaces and completeness of mappings and spaces*, Gen. Top. Appl. 1 (1971), 85-100.
- [W₂] _____, *Base of countable order theory and some generalizations*, in Proc. Univ. of Houston Point Set Topology Conf. 1971, Houston, Tex., 1972, 76-95.
- [W₃] _____, *On spaces whose diagonal is a set of interior condensation*, Notices Amer. Math. Soc. 19 (1972), A-567.
- [W₄] _____, *Complete mappings in base of countable order theory*, in Set Theoretic Topology, Academic Press, 1977, 383-412.
- [WW₁] _____ and J. M. Worrell, Jr., *Open continuous mappings of spaces having bases of countable order*, Duke Math. J. 34 (1967), 255-272.
- [WW₂] _____, *Topological completeness of first countable Hausdorff spaces I*, Fund. Math. 75 (1972), 209-222.
- [WW₃] _____, *Primitive structures in general topology*, in Studies in Topology, Academic Press, New York,

- 1975, 581-599.
- [WW₄] H. H. Wicke and J. M. Worrell, Jr., *A characterization of spaces having bases of countable order in terms of primitive bases*, Can. J. Math. 27 (1975), 1100-1109.
- [WW₅] _____, *The concept of a θ -refinable embedding*, Gen. Top. Appl. 6 (1976), 167-181.
- [WW₆] _____, *Topological completeness of first countable Hausdorff spaces II*, Fund. Math. 91 (1976), 11-27.
- [WW₇] _____, *The local-implies-global characteristic of primitive sequences*, in Proceedings Memphis State University Topology Conference, Marcel Dekker, N.Y., 1976, 269-282.
- [WW₈] _____, *Scattered spaces of point-countable type*, 1976 Prague Topology Conference, to appear.
- [WoW] J. M. Worrell, Jr. and H. H. Wicke, *Characterizations of developable topological spaces*, Can. J. Math. 17 (1965), 820-830.

Ohio University

Athens, Ohio 45701