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ON THE EXISTENCE OF ARCS IN RATIONAL CURVES

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A *rational continuum* is a compact connected metric space which has a basis of open sets with countable boundaries. Rational curves which contain no arcs have interested topologists for a long time. In 1912, Z. Janiszewski (see [2]) constructed an arclike rational curve which contains no arcs. In 1930, G. T. Whyburn [6] constructed a rational plane continuum every subcontinuum of which separates the plane.

If A is a subset of a topological space X let A' denote the derived set of A . Let $A^{(0)} = A$ and by transfinite induction define $A^{(\alpha)}$ for each ordinal α , by $A^{(\alpha+1)} = (A^{(\alpha)})'$ and $A^{(\lambda)} = \bigcap \{A^{(\alpha)} \mid \alpha < \lambda\}$ for each limit ordinal λ . If C is a compact, countable subset of a metric space then there exists a countable ordinal α such that $C^{(\alpha)} = \emptyset$. We denote the smallest such ordinal α by $ttyp(C)$, the topological type of C . If X is a rational continuum and $x \in X$ we define the *rim-type of X at x* by $\text{rimt}_x(X) = \alpha$, where α is the smallest ordinal number such that X has a neighbourhood basis at x of open sets $\{U_i\}_{i \in \mathbb{N}}$ such that $ttyp(\text{Bd}(U_i)) \leq \alpha$ for each $i \in \mathbb{N}$. Then $\text{rimt}_x(X)$ is a countable ordinal. Finally we define the *rim-type of X* by

$$\text{rimt}(X) = \sup\{\text{rimt}_x(X) \mid x \in X\}$$

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It is well-known (see [3], p. 290) that the rim-type of a rational continuum is a countable ordinal.

The examples of Janiszewski and Whyburn which are mentioned above are both of rim-type ω . A. Lelek asked in [4] whether every rational curve of finite rim-type contains an arc. In [5] Lelek and Mohler answered Lelek's question in the affirmative for the case of hereditarily unicoherent continua. It is our purpose in this note to announce an affirmative solution to Lelek's question in its most general form. A complete version of this note will appear elsewhere.

The following result is well-known (see [3], p. 216):

Theorem 1. If X is an irreducible, hereditarily decomposable continuum, then there exists a finest monotone mapping of X onto the unit interval $[0,1]$. The point-inverses under this mapping are nowhere dense subcontinua of X and are called the tranches of X .

Theorem 2. Every rational continuum of finite rim-type contains an arc.

Outline of proof. The proof is by induction on the rim-type of X . If the rim-type of X is 1, then X has a basis of open sets with finite boundaries. It is well-known that in this case X is locally connected, and hence, X contains an arc. Suppose that each rational continuum of rim-type $\leq n - 1$ contains an arc.

Just suppose that X is a rational continuum of rim-type n such that X contains no arc. Then every non-degenerate subcontinuum of X has rim-type n . We may suppose, without loss of generality, that X is an irreducible continuum. Let

$\pi: X \rightarrow [0,1]$ be a finest monotone mapping of X onto $[0,1]$.

Since X is not an arc we may suppose $\pi^{-1}(1)$ is non-degenerate.

Let $\mathcal{U} = \{U_i, \dots\}$ be a countable basis of open sets for X whose boundaries are pairwise disjoint and have topological type $\leq n$. We may also suppose that if x is a boundary point of U_i , then x is in the closure of the complement of the closure of U_i .

We use the fact that every subset of X which contains $\pi^{-1}([0,1])$ is connected to kill the local separating points of the boundary of U_i which are in $\pi^{-1}(1)$. We get, by using induction and inverse limit techniques, a compactification Y of $X \setminus \pi^{-1}(1)$ that is larger than X and such that if $g: Y \rightarrow X$ is the natural map, then g is at most 2 to 1, g is 1 to 1, except on a countable set, and $g^{-1}(\pi^{-1}(1))$ is a continuum of rim-type $\leq n - 1$. It follows from the induction hypothesis that Y contains an arc. Since g is at most 2 to 1 X also contains an arc.

B. B. Epps, Jr., in [1], has constructed for each positive integer n an example of an arclike rational continuum X of rim-type n such that for each subcontinuum Y of X $\text{rimt}(Y) \in \{1, n\}$. Hence, our result is the best possible.

A continuum X is said to be *atriodic* if there do not exist three continua A_1, A_2 and A_3 in X such that $A_1 \cap A_2 \cap A_3 \neq \emptyset$ and $A_i \not\subset A_j \cup A_k$ for $i \notin \{j, k\}$. The above techniques can be used to prove the following theorem concerning the structure of atriodic continua.

Theorem 3. If X is an atriodic, irreducible, metric continuum that admits a finest monotone map $\pi: X \rightarrow [0,1]$

onto $[0,1]$ such that $\pi^{-1}(1)$ is non-degenerate, then there exists an irreducible, atriodic metric continuum Y and a map g of Y onto X such that g is one to one except on a countable set in $\pi^{-1}(1)$, g is at most 2 to 1, $\pi \circ g$ is a finest monotone map of Y onto $[0,1]$ and $g^{-1} \circ \pi^{-1}(1)$ contains no local separating points of Y .

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