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SUCH THAT P^2 IS COMPLETELY
NORMAL

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A COMPACT NONMETRIZABLE SPACE P SUCH THAT P^2 IS COMPLETELY NORMAL

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In 1948, M. Katětov showed [2] that if P is a compact space such that $P \times P \times P$ is completely normal, then P is metrizable. [Throughout this announcement, the word "space" will always refer to a Hausdorff space.] Katětov then remarked that he did not know whether complete normality of $P \times P$ was enough to give metrizability. I will now give an example of a compact, nonmetrizable space P whose square, assuming Martin's axiom and the negation of the continuum hypothesis [abbreviated $MA + \neg CH$] is completely normal. I will also indicate why a "naive" example of such a space, if it exists, will be hard to come by.

1. The Example

Let $I = [0, 1]$. Let $A \in I$ be of cardinality \aleph_1 . Replace each a of A by two points, a^+ and a^- , with the convention that $a^- < a^+$; otherwise we keep the usual order of all of $(I-A) \cup A^+ \cup A^- = P$. Since P is complete in the order topology, it is a compact space, and nonmetrizable because it does not have a countable base: every base must include an open set with a^+ as its least element, and one with a^- as its greatest, for all $a \in A$.

2. Preliminary Lemmas

The following oft-used lemma is utilized in the proof:

Lemma 1. Let H and K be subsets of a space X . H and

K can be put into disjoint open subsets whenever there exist countable collections of open sets U_n and V_n so that $H \subset \bigcup_{n=1}^{\infty} U_n$, $K \subset \bigcup_{n=1}^{\infty} V_n$ and $\overline{U}_n \cap K = \phi$ and $\overline{V}_n \cap H = \phi$ for all n .

We also use:

Lemma 2. [3] [MA + \neg CH] Let Y be a metric space of cardinality \aleph_1 . Every subset of Y is an F_σ .

We also note that $(A^+)^2$ is homeomorphic to a subspace of the Sorgenfrey plane.

Lemma 3. [1] [MA + \neg CH] If X is a subspace of the Sorgenfrey plane, of cardinal \aleph_1 , then X is normal.

Lemma 4. The space $P^2 - (A^+ \cup A^-)^2$ is hereditarily Lindelöf.

Now, although P^2 is the union of the perfectly normal (under MA + \neg CH) subspaces $P^2 - (A^+ \cup A^-)^2$, $(A^+)^2$, $(A^-)^2$, $A^+ \times A^-$, and $A^- \times A^+$, the whole space is not perfectly normal; for example, the diagonal is not a G_δ . But under MA + \neg CH it is completely normal.

3. Outline of the Proof

Let H and K be subsets of P^2 such that $\overline{H} \cap K = \phi$, $H \cap \overline{K} = \phi$. The objective is to find countable collections $\{U_n\}_{n=1}^{\infty}$ and $\{V_n\}_{n=1}^{\infty}$ of open subsets of P^2 as in Lemma 1. Using Lemma 4 and regularity of P^2 , we can get $H - (A^+ \cup A^-)^2$ into a countable union of open sets whose closure misses K . By various symmetry arguments, the proof then boils down to showing that there exist countably many open sets whose closures miss K and whose union contains $H \cap (A^+)^2 = H_1$.

With the help of Lemma 3, we can get H_1 into an open subset of P^2 whose closure misses $K \cap (A^+)^2$. We will choose all our open sets to be contained in this one. By Lemma 2 [there is a coarser metric topology on $(A^+)^2$] we can let $H_1 = \bigcup_{n=1}^{\infty} F_n$ where each F_n is closed in $(A^+)^2$, and each $x \in F_n$ has a basic first-quadrant neighborhood whose closure misses K and which is a square $1/n$ on a side. For each F_n we cut up P^2 into countably many clopen squares $<1/n$ on a side. It is enough to take care of the points of F_n which lie in any one square S . Attach the basic $1/n$ -neighborhoods to these points, trimming off the parts sticking out of S . The only possible points of K in the closure of the resulting open set lie on a graph which can be thought of as a monotone function. Moreover, points of K can only lie along straight line segments of the graph, of which there are countably many.

Let E_n stand for the one-sided limit points of $F_n \cap S$ on the graph. We use Lemma 2 again to show $F_n \cap S = \bigcup_{m=1}^{\infty} C_{nm}$ where each C_{nm} is closed in the relative *Euclidean* topology of $E_n \cup (F_n \cap S)$. If we attach the basic $1/n$ -neighborhoods to the points of C_{nm} and intersect with S , it turns out that the closure of the resulting set misses K .

4. What Happens if $2^{\aleph_0} < 2^{\aleph_1}$

The space P^2 is *not* completely normal in any model of set theory where $2^{\aleph_0} < 2^{\aleph_1}$. This is because P^2 has the uncountable discrete subspace $\{(a^+, a^-) \mid a \in A\}$ and is separable, and it is well known that:

Lemma 5. $[2^{\aleph_0} < 2^{\aleph_1}]$ Every discrete subspace of a separable, completely normal space is countable.

This lemma quickly involves us in a famous pair of problems of general topology: whether there exists an S-space (a regular, hereditarily separable space which is not hereditarily Lindelöf) or an L-space (a regular, hereditarily Lindelöf space which is not hereditarily separable). Such spaces have been constructed in some models of set theory; but, in particular, no one has constructed such spaces assuming *only* $2^{\aleph_0} < 2^{\aleph_1}$.

But from Lemma 5 it is only a short step to:

Theorem [$2^{\aleph_0} < 2^{\aleph_1}$] *If X is a compact, nonmetrizable space such that X^2 is completely normal, then at least one of the following is true:*

1. *X is an L-space*
2. *X^2 is an S-space*
3. *X^2 contains both an S-space and an L-space.*

If we combine this theorem with the result that $MA + \neg CH$ implies every compact space of countable spread is hereditarily separable, we see that a "naive" example (one whose basic cardinal invariants like density, spread, and hereditary Lindelöf degree do not vary with the model of ZFC used) would have a hereditarily separable square which is not hereditarily Lindelöf (because of not having a G_δ -diagonal). Such a compact space has not yet been constructed in *any* model of set theory!

Even if we restrict ourselves to $2^{\aleph_0} < 2^{\aleph_1}$, we are out of reasonable candidates: Mary Ellen Rudin showed, after I came up with these results, that the square of a Souslin line is never completely normal.

References

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- [3] J. Shinoda, *Some consequences of Martin's Axiom and the negation of the Continuum Hypothesis*, Nagoya Math. J. 49 (1973), 117-125.

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