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1. Introduction

A *continuum* is a compact connected metric space. The main results of this paper (theorems 3.7 and 3.8) are a pair of characterizations of indecomposability for tree-like continua.[1] One of the most useful characterizations of indecomposability is the Cook-Ingram theorem (theorem 1.4 below) which relates indecomposability to the properties of a defining sequence of open covers for the space. This type of theorem is especially useful for tree-like continua, whose very definition involves properties of covers of the space. Our theorems are strengthened versions of the Cook-Ingram theorem for tree-like continua. They say roughly that subchains of refining covers can be made to play the role of the entire refining cover in the Cook-Ingram theorem.

The paper is in five sections. The remainder of the present section introduces some terminology for covers and states the Cook-Ingram theorem. Section 2 is devoted to the proof of a proposition (2.5) on the existence of certain special refinements for open covers of an indecomposable

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[1]For the basic facts about indecomposable continua the reader is referred to [7], chapter 3, section 8, or [9], section 48, paragraphs V and VI. (See the bibliography at the end of the paper.) The necessary background material concerning tree-like continua is summarized in the body of the paper (see section 3).

continuum. A refinement of this proposition (3.6) yields our two main theorems, which are the main concern of section 3. Section 4 is taken up with applications while section 5 covers some remarks and questions.

Definition 1.1. Let $\mathcal{K}$ be a collection of sets. A pair of subcollections, $\mathcal{K}_1$ and $\mathcal{K}_2$, of $\mathcal{K}$ is called a *separation* of $\mathcal{K}$ if $\cup\mathcal{K}_1 \neq \emptyset \neq \cup\mathcal{K}_2$, $\mathcal{K}_1 \cup \mathcal{K}_2 = \mathcal{K}$, and $(\cup\mathcal{K}_1) \cap (\cup\mathcal{K}_2) = \emptyset$. $\mathcal{K}$ is said to be *coherent* if it does not admit a separation. This is equivalent to saying that the nerve of $\mathcal{K}$ is connected. Note that any open cover of a connected space is coherent.

Definition 1.2. Let X be a topological space and let $\mathcal{U}$ be an open cover of X. $\mathcal{U}$ is said to be *irreducible* if each set $U \in \mathcal{U}$ contains a point x ($\in$ X) which lies in no other set $V \in \mathcal{U}$. Such an x is called a *point of irreducibility* of U. Note that any finite cover of a topological space contains an irreducible subcover.

Definition 1.3. Let X be a continuum and let $\mathcal{U}_1, \mathcal{U}_2, \cdots$ be a sequence of open covers of X. $\mathcal{U}_1, \mathcal{U}_2, \cdots$ will be called a *defining sequence* if

i) Each $\mathcal{U}_n$ is an irreducible cover of X,

ii) Each $\mathcal{U}_n$ is a coherent collection (this condition is redundant),

iii) $\lim_{n\to\infty}$ mesh $(\mathcal{U}_n) = 0$, and

iv) For every n, $\mathcal{U}_{n+1}$ is a strong refinement of $\mathcal{U}_n$ (that is, the closures of elements of $\mathcal{U}_{n+1}$ refine $\mathcal{U}_n$).

Every continuum admits a defining sequence of open covers. If $\mathcal{U}_1, \mathcal{U}_2, \cdots$ is a sequence of open covers of X satisfying conditions i)-iii) above and

iv') For every n, $\mathcal{U}_{n+1}$ is a refinement of $\mathcal{U}_n$,

then $\mathcal{U}_1, \mathcal{U}_2, \cdots$ will be called a *weak defining sequence* for X.

Theorem 1.4 (Cook-Ingram). *A continuum* X *is indecomposable if and only if* X *admits a defining sequence of covers* $\mathcal{U}_1, \mathcal{U}_2, \cdots$ *such that for each* n*, there is a* $k > n$ *such that if* $\mathcal{U}_k$ *is the union of two coherent subcollections,* $\mathcal{U}_{k,1}$ *and* $\mathcal{U}_{k,2}$*, then either* $U \cap (\cup\mathcal{U}_{k,1}) \neq \emptyset$ *for every* $U \in \mathcal{U}_n$ *or* $U \cap (\cup\mathcal{U}_{k,2}) \neq \emptyset$ *for every* $U \in \mathcal{U}_n$.

Proof. See [4].

Definition 1.5. Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two collections of non-empty sets. $\mathcal{K}_1$ will be said to *span* $\mathcal{K}_2$ if each element of $\mathcal{K}_2$ contains an element of $\mathcal{K}_1$.

Definition 1.6. Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two coherent collections of non-empty sets. $\mathcal{K}_1$ will be said to satisfy the *Cook-Ingram condition* in $\mathcal{K}_2$ if whenever $\mathcal{K}_1$ is the union of two coherent subcollections, $\mathcal{K}_{1,1}$ and $\mathcal{K}_{1,2}$, either $\mathcal{K}_{1,1}$ spans $\mathcal{K}_2$ or $\mathcal{K}_{1,2}$ spans $\mathcal{K}_2$.
The above condition is stronger than the condition on $\mathcal{U}_k$ in 1.4. It is adopted because it simplifies the arguments below and its implications turn out to be the same in our setting. We finish this section with the introduction of a strong form of the irreducibility condition for covers.

Definition 1.7. Let $\mathcal{U}$ be an open cover of the space X and let $U \in \mathcal{U}$. A point $x \in U$ will be called an *essential point* of U if x is not an element of the closure of any other member of $\mathcal{U}$. The set of all essential points of U will be denoted by $E(U,\mathcal{U})$. $\mathcal{U}$ is called an *essential cover* of X if $E(U,\mathcal{U}) \neq \emptyset$ for every $U \in \mathcal{U}$.

Lemma 1.8. *Let* $\mathcal{U} = \{U_\alpha : \alpha \in \Gamma\}$ *be an irreducible (finite) open cover of the continuum* X. *Then* $\mathcal{U}$ *admits an essential refinement.*

Proof. By [5], theorem 6.1, p. 152 $\mathcal{U}$ admits a refinement $\mathcal{V} = \{V_\alpha : \alpha \in \Gamma\}$ such that $Cl(V_\alpha) \subset U_\alpha$ for each $\alpha \in \Gamma$. It follows that if x is a point of irreducibility of U_α, it will be an essential point of V_α.

2. A Structure Theorem for Indecomposable Continua

This section is devoted to the proof of 2.5, which will suffice to characterize indecomposability for tree-like continua.

Definition 2.1. A finite collection $\mathcal{C}$ of sets is called a *chain* if and only if the elements of $\mathcal{C}$ can be numbered $C_0, C_1, \cdots, C_n$ in such a way that $C_i \cap C_j \neq \emptyset$ if and only if $|i - j| \leq 1$. Note that chains are coherent and that a subfamily of a chain is coherent if and only if it is a chain. The elements of $\mathcal{C}$ are called *links*. C_0 and C_n are called *end links* of $\mathcal{C}$. If $\mathcal{C}$ satisfies the above conditions except that $C_0 \cap C_n \neq \emptyset$, then $\mathcal{C}$ is called a *circular chain*.
The proof of the following lemma is left to the reader:

Lemma 2.2. *Let* $\mathcal{K}$ *be a coherent collection of non-empty sets and let* U, V $\in \mathcal{K}$. *Then* $\mathcal{K}$ *contains a chain whose end links are* U *and* V.

Lemma 2.3. *Let* X *be an indecomposable continuum and* *let $\mathcal{U}$ be any finite collection of non-empty open subsets of* X. *Then there exist disjoint subcontinua,* C_1 *and* C_2*, of* X*, lying in distinct composants, such that* $C_i \cap U \neq \emptyset$ *for each* $U \in \mathcal{U}$ *and for* $i = 1,2$.

Proof. Let p_1 and p_2 be points of X lying in distinct composants. Since composants are dense, for each $U \in \mathcal{U}$ and for $i = 1,2$, there is a proper subcontinuum $C(i,U)$ of X containing p_i and meeting the open set U. Let $C_1 = \cup\{C(1,U): U \in \mathcal{U}\}$ and $C_2 = \cup\{C(2,U): U \in \mathcal{U}\}$. Since composants are disjoint, $C_1 \cap C_2 = \emptyset$.

Theorem 2.4. *Let* X *be an indecomposable continuum and let $\mathcal{U}$ be a finite irreducible open cover of* X. *Then $\mathcal{U}$ admits an essential refinement, $\mathcal{U}^+$, containing a chain $\mathcal{C}'$ which spans $\mathcal{U}$.*

Proof.[2] Let X and $\mathcal{U}$ be as above and suppose that no such $\mathcal{U}^+$ exists. Let $\mathcal{V}$ be an essential refinement of $\mathcal{U}$, constructed as in Lemma 1.8. Then $\mathcal{V}$ spans $\mathcal{U}$. Let n be the cardinality of $\mathcal{V}$ (= card ($\mathcal{U}$)) and let k be the cardinality of a subfamily $\mathcal{B} = \{V_1, \cdots, V_k\}$ of $\mathcal{V}$ which is maximal with respect to the property of being spanned by a subchain of some essential refinement of $\mathcal{V}$. By hypothesis $k < n$ and since $\mathcal{V}$ is coherent and refines itself, Lemma 2.2 implies that $k \geq 2$. Let $\mathcal{U}'$ be an essential refinement of $\mathcal{V}$ containing a chain $\mathcal{C} = \{U_0, \cdots, U_r\}$ which spans $\mathcal{B}$. Let $V_{k+1} \in \mathcal{V} - \mathcal{B}$.

[2]A simpler proof of this theorem can be given using a result of Cook [3] (p. 40, theorem 6). The conclusion will also be stronger, since one need only assume that X is irreducible. The authors are indebted to the referee for this observation. We retain the proof given here because some details of the construction will be needed later. See footnote 3.

Let C_1 and C_2 be disjoint subcontinua of X, lying in distinct composants of X, such that for i = 1,2:

i) $C_i \cap E(U,\mathcal{U}') \neq \emptyset$ for every $U \in \mathcal{U}'$ and

ii) $C_i \cap (\cap\mathcal{W}) \neq \emptyset$ for every $\mathcal{W} \subset \mathcal{U}'$ such that $\cap\mathcal{W} \neq \emptyset$.

Let e be an essential point of U_r which is in neither C_1 nor C_2 and let W be a neighborhood of e whose closure is contained in $E(U_r,\mathcal{U}') - (C_1 \cup C_2)$. Since C_1 and C_2 lie in distinct composants, they lie in distinct components of X - W. Thus there exist compact sets K_1 and K_2 such that $K_1 \cup K_2 = X - W$, $K_1 \cap K_2 = \emptyset$ and $C_i \subset K_i$ for i = 1,2. Let V be a neighborhood of e whose closure is contained in W. Then by the normality of X, there exist open subsets P and Q of X such that

iii) $K_1 \subset P$ and $K_2 \subset Q$,

iv) $P \cap Q = \emptyset$ and

v) $(P \cup Q) \cap Cl(V) = \emptyset$.

Note that $X = P \cup Q \cup W$.

Let $\mathcal{S}^P = \{U \cap P: U \in \mathcal{U}'\}$ and $\mathcal{S}^Q = \{U \cap Q: U \in \mathcal{U}'\}$. Let $\mathcal{S} = \mathcal{S}^P \cup \mathcal{S}^Q \cup \{W\}$. Since $C_1 \subset P = \cup\mathcal{S}^P$ and $C_2 \subset Q = \cup\mathcal{S}^Q$, i) and iv) imply that $E(U \cap P,\mathcal{S}) \neq \emptyset$ and $E(U \cap Q,\mathcal{S}) \neq \emptyset$ for every $U \in \mathcal{U}' - \{U_r\}$. Since $C_i \cap Cl(W) = \emptyset$ for i = 1,2, i) and iv) imply that $E(U_r \cap P,\mathcal{S}) \neq \emptyset \neq E(U_r \cap Q,\mathcal{S})$. Finally, v) implies that $e \in E(W,\mathcal{S})$. Thus $\mathcal{S}$ is an essential refinement of $\mathcal{U}'$ and hence of $\mathcal{V}$. We now show that $\mathcal{S}$ contains a chain spanning the family $\mathcal{B} \cup \{V_{k+1}\}$, contradicting the maximality of $\mathcal{B}$. This will complete the proof.

Condition ii) above ensures that if $G, H \in \mathcal{U}'$ and $G \cap H \neq \emptyset$, then $(G \cap P) \cap (H \cap P) \neq \emptyset \neq (G \cap Q) \cap (H \cap Q)$. Thus $\mathcal{S}^P$ and $\mathcal{S}^Q$ are both coherent collections since $\mathcal{U}'$

is.[3] Also, $\{U_0 \cap P, U_1 \cap P, \cdots, U_r \cap P\}$ is a chain in $\mathcal{S}^P$ since $\mathcal{C}$ is a chain in $\mathcal{U}'$. Since $\mathcal{V}$ is irreducible and $\mathcal{U}'$ refines $\mathcal{V}$, there must be a $U \in \mathcal{U}'$ such that $U \subset V_{k+1}$. Since $\mathcal{U}'$ is coherent, there must be a chain $\{U_r, U_{r+1}, \cdots, U_m\}$ in $\mathcal{U}'$ whose end links are U_r and U respectively. ii) implies that $\{U_r \cap Q, U_{r+1} \cap Q, \cdots, U_m \cap Q\}$ is a chain in $\mathcal{S}^Q$. Finally, the coherence of $\mathcal{S}$ implies that $(U_r \cap P) \cap W \neq \emptyset \neq (U_r \cap Q) \cap W$. Thus $\{U_0 \cap P, U_1 \cap P, \cdots, U_r \cap P, W, U_r \cap Q, U_{r+1} \cap Q, \cdots, U_m \cap Q\}$ is a chain in $\mathcal{S}$ which spans $\mathcal{B} \cup \{V_{k+1}\}$.

Theorem 2.5. *Let* X *be an indecomposable continuum and let* $\mathcal{U}$ *be a finite irreducible open cover of* X. *Then* $\mathcal{U}$ *admits an essential refinement* $\mathcal{U}^{++}$ *containing a chain* $\mathcal{C}$ *which satisfies the Cook-Ingram condition in* $\mathcal{U}$.

Proof. Let $\mathcal{U}$, $\mathcal{U}^+$ and $\mathcal{C}' = \{U_0, U_1, \cdots, U_n\}$ be as in the statement of theorem 2.4. Now construct the refinement $\mathcal{U}^{++}$ from $\mathcal{U}^+$ just as $\mathcal{S}$ was constructed from $\mathcal{U}'$ in the proof of 2.4, with $\mathcal{C}'$ playing the role of $\mathcal{C}$ and U_n the role of U_r. Let $\mathcal{C} = \{U_0 \cap P, U_1 \cap P, \cdots, U_n \cap P, W, U_n \cap Q, U_{n-1} \cap Q, \cdots, U_0 \cap Q\}$. Then $\mathcal{C}$ is the union of chains $\mathcal{C}_1 = \{U_0 \cap P, \cdots, U_n \cap P, W\}$ and $\mathcal{C}_2 = \{U_0 \cap Q, \cdots, U_n \cap Q, W\}$ both of which span $\mathcal{S}$ and which have only the link W in common. It is not difficult to show that $\mathcal{C}$ must then have the Cook-Ingram property in $\mathcal{U}$.

3. The Main Theorems

Theorem 2.5 contains all of the difficult material required for the proof of the main theorems (3.7 and 3.8 below).

[3]In fact, if $\mathcal{U}'$ is a tree chain, then $\mathcal{S}$ is a tree chain. This fact will be needed in the proof of 3.6 below.

All that remains are some special definitions and lemmas pertaining to tree-like continua.

Definition 3.1. A finite collection $\mathcal{J}$ of sets is called a *tree-chain* if i) $\mathcal{J}$ is coherent, ii) no three elements of $\mathcal{J}$ have a point in common and iii) $\mathcal{J}$ contains no circular chains. The elements of $\mathcal{J}$ are called *links*.

Definition 3.2. A continuum X is said to be *tree-like* if and only if for every $\varepsilon > 0$ X admits an open cover of mesh $\leq \varepsilon$ which is a tree-chain. The following lemma summarizes a collection of facts about tree-chains and tree-like continua which will be assumed in what follows. The proofs are left to the reader.

Lemma 3.3. *Let* $\mathcal{J}$ *be a tree-chain. Then*

i) Every coherent subcollection of $\mathcal{J}$ *is a tree-chain.*

ii) If L_1 *and* L_2 *are distinct links of* $\mathcal{J}$, *then there is a unique chain in* $\mathcal{J}$ *whose end links are* L_1 *and* L_2. *This chain will be denoted by* $\underline{\mathcal{J}[L_1,L_2]}$ *or if no confusion seems likely simply* $\underline{[L_1,L_2]}$.

iii) If L_1 *and* L_2 *are distinct links of* $\mathcal{J}$ *and* L *is a non-end link of* $[L_1,L_2]$, *then* $\mathcal{J}-\{L\}$ *admits a separation* $\mathcal{P}_1,\mathcal{P}_2$ *such that* $L_1 \in \mathcal{P}_1$ *and* $L_2 \in \mathcal{P}_2$.

iv) If $\mathcal{K}$ *is a coherent subcollection of* $\mathcal{J}$ *and* $\mathcal{C}$ *is a subchain of* $\mathcal{J}$, *then* $\mathcal{K} \cap \mathcal{C}$ *is a subchain of* $\mathcal{J}$.

v) If X *is a tree-like continuum, then there is a defining sequence* $\mathcal{J}_1,\mathcal{J}_2,\cdots$ *for* X, *each term of which is a tree-chain.*

For the rest of the paper, whenever the phrases *defining sequence* and *weak defining sequence* are used in reference to a tree-like continuum, it will be assumed that each term of the sequence in question is a tree-chain.

Lemma 3.4. *Let $\mathcal{J}$ and $\mathcal{J}'$ be tree-chains and suppose that $\mathcal{J}'$ refines $\mathcal{J}$. Let L_1 and L_2 be distinct links of $\mathcal{J}$. If L_1' and L_2' are links of $\mathcal{J}'$ such that $L_1' \subset L_1$ and $L_2' \subset L_2$, then $\mathcal{J}'[L_1',L_2']$ spans $\mathcal{J}[L_1,L_2]$.*

Proof. Suppose not and let L be a link of $[L_1,L_2]$ which contains no link of $[L_1',L_2']$. L is not an end link of $[L_1,L_2]$, so by 3.3 iii) there is a separation $\mathcal{P}_1, \mathcal{P}_2$ of $\mathcal{J}$-{L} such that $L_1 \in \mathcal{P}_1$ and $L_2 \in \mathcal{P}_2$. By assumption each link of $[L_1',L_2']$ is contained in some link of $\mathcal{J}-\{L\} = \mathcal{P}_1 \cup \mathcal{P}_2$. Let $\mathcal{L}_1 = \{L' \in [L_1',L_2']: L' \subset \cup \mathcal{P}_1\}$ and $\mathcal{L}_2 = \{L' \in [L_1',L_2']: L' \subset \cup \mathcal{P}_2\}$. Then $\mathcal{L}_1, \mathcal{L}_2$ is a separation of $[L_1',L_2']$, contradicting the coherence of this family.

Lemma 3.5. *Let $\mathcal{J}_1$, $\mathcal{J}_2$ and $\mathcal{J}_3$ be irreducible, open tree-chain covers of the continuum* X *such that $\mathcal{J}_1$ refines $\mathcal{J}_2$ and $\mathcal{J}_2$ refines $\mathcal{J}_3$. If $\mathcal{J}_2$ contains a chain satisfying the Cook-Ingram condition in $\mathcal{J}_3$, then $\mathcal{J}_1$ also contains such a chain.*

Proof. Let $[L_1,L_2]$ be a subchain of $\mathcal{J}_2$ satisfying the Cook-Ingram condition in $\mathcal{J}_3$. Since $\mathcal{J}_2$ is irreducible and $\mathcal{J}_1$ covers X and refines $\mathcal{J}_2$, there must be links L_1' and L_2' of $\mathcal{J}_1$ such that $L_1' \subset L_1$ and $L_2' \subset L_2$. We claim that $[L_1',L_2']$ satisfies the Cook-Ingram condition in $\mathcal{J}_3$.

Suppose that $[L_1',L_2']$ is the union of two subchains $[L_1',L_3']$ and $[L_4',L_2']$. Let L_3 be the "rightmost" link of $[L_1,L_2]$ which contains a link of $[L_1',L_3']$ and let L_4 be the "next" link of $[L_1,L_2]$. Then by lemma 3.4 $[L_1',L_3']$ spans

$[L_1,L_3]$ and $[L_4',L_2']$ spans $[L_4,L_2]$. But either $[L_1,L_3]$ or $[L_4,L_2]$ spans $\mathcal{J}_3$. Thus either $[L_1',L_3']$ or $[L_4',L']$ spans $\mathcal{J}_3$.

Theorem 3.6. *Let* X *be an indecomposable tree-like continuum and let* $\mathcal{J}$ *be an irreducible open tree-chain cover of* X. *Then there exists an essential open tree-chain cover of* X *which refines* $\mathcal{J}$ *and contains a chain satisfying the Cook-Ingram condition in* $\mathcal{J}$.

Proof. This theorem is proved in exactly the same way as 2.4 and 2.5. The only problem is to make sure that the various refinements which appear in the proof are all tree-chains. This can be done by hypothesis except for the covers $\mathcal{S}$ and $\mathcal{U}^{++}$. These will be tree-chains by construction (see footnote 3).

Theorem 3.7. *Let* X *be a tree-like continuum. Then* X *is indecomposable if and only if given any weak defining sequence* $\mathcal{J}_1,\mathcal{J}_2,\cdots$ *for* X *and any* n, *there exists an* $m \geq n$ *such that for all* $k \geq m$, $\mathcal{J}_k$ *contains a chain satisfying the Cook-Ingram condition in* $\mathcal{J}_n$.

Proof. Suppose that X is indecomposable and let $\mathcal{J}_1,\mathcal{J}_2,\cdots$ be a weak defining sequence for X. Let n be given. Let $\mathcal{J}'$ be an essential open tree-chain cover of X which refines $\mathcal{J}_n$ and contains a chain satisfying the Cook-Ingram condition in $\mathcal{J}_n$. Since $\lim_{j\to\infty}$ mesh $(\mathcal{J}_j) = 0$, there is an $m \geq n$ such that for all $k \geq m$, $\mathcal{J}_k$ refines $\mathcal{J}'$ (eventually mesh $(\mathcal{J}_j)$ is smaller than the Lebesgue number of $\mathcal{J}'$). Lemma 3.5 implies that all such $\mathcal{J}_k$ will contain a chain satisfying the Cook-Ingram condition in $\mathcal{J}_n$. For the other half of the proof, see the proof of the following theorem.

Theorem 3.8. *Let* X *be a tree-like continuum. Then* X *is indecomposable if and only if there exists a weak defining sequence* $\mathcal{J}_1, \mathcal{J}_2, \cdots$ *for* S *such that given any* n, *there is a* $k \geq n$ *such that* $\mathcal{J}_k$ *contains a chain satisfying the Cook-Ingram condition in* $\mathcal{J}_n$.

Proof. Let $\mathcal{J}_1, \mathcal{J}_2, \cdots$ be a weak defining sequence for X satisfying the above condition. Let $\mathcal{S}_1, \mathcal{S}_2, \cdots$ be a defining sequence for X (see 3.3.v). We will show that $\mathcal{S}_1, \mathcal{S}_2, \cdots$ satisfies the hypothesis of the Cook-Ingram theorem. Let n be given and choose an $n' > n$ such that $\mathcal{J}_{n'}$ refines $\mathcal{S}_n$. Choose a $k' \geq n'$ such that $\mathcal{J}_{k'}$ contains a chain, $\mathcal{C}$, satisfying the Cook-Ingram condition in $\mathcal{J}_{n'}$. Since $\mathcal{S}_n$ is irreducible, $\mathcal{J}_{n'}$ spans $\mathcal{S}_n$. Thus $\mathcal{C}$ also satisfies the Cook-Ingram condition in $\mathcal{S}_n$. Note further that $\mathcal{J}_{k'}$ refines $\mathcal{S}_n$.

Now choose a $k \geq k'$ such that $\mathcal{S}_k$ refines $\mathcal{J}_{k'}$. Then by 3.5 $\mathcal{S}_k$ contains a chain $\mathcal{L}$ satisfying the Cook-Ingram condition in $\mathcal{S}_n$. Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two coherent subcollections of $\mathcal{S}_k$ such that $\mathcal{K}_1 \cup \mathcal{K}_2 = \mathcal{S}_k$. Let $\mathcal{L}_1 = \mathcal{L} \cap \mathcal{K}_1$ and $\mathcal{L}_2 = \mathcal{L} \cap \mathcal{K}_2$. Then one of these two chains (see 3.3.iv), say $\mathcal{L}_1$, spans $\mathcal{S}_n$. Thus $\mathcal{K}_1$ meets every link of $\mathcal{S}_n$. This completes the proof of one half of the theorem. For the other half see the proof of the previous theorem.

Theorem 3.6 can be rephrased in the following useful manner:

Theorem 3.9. *Let* X *be a tree-like continuum. Then* X *is indecomposable if and only if given any irreducible tree-chain open cover* $\mathcal{J}$ *of* X, *there is a* $\delta > 0$ *such that if* $\mathcal{J}'$ *is an irreducible tree-chain open cover of* X *of mesh* $\leq \delta$, *then*

$\mathcal{T}'$ contains a chain $\mathcal{C}$ with the property that if $\mathcal{C}_1$ and $\mathcal{C}_2$ are chains whose union is $\mathcal{C}$, then either $\mathcal{C}_1$ or $\mathcal{C}_2$ spans $\mathcal{T}$.

Proof. Let X be an indecomposable tree-like continuum and let $\mathcal{T}$ be an irreducible open tree-chain cover of X. Then $\mathcal{T}$ admits an irreducible refinement $\mathcal{T}_0$ which contains a chain $\mathcal{C}$ satisfying the above condition (see theorem 3.6). Choose δ such that if $\mathcal{T}'$ is a cover of X of mesh $\leq \delta$, then $\mathcal{T}'$ refines $\mathcal{T}$. It follows that any such $\mathcal{T}'$ (which is an open tree-chain cover) contains a chain satisfying the desired condition (see lemma 3.5).

Now suppose that X satisfies the condition of the theorem. Let $\mathcal{T}_1, \mathcal{T}_2, \cdots$ be a defining sequence for X and let n be given. Choose k > n such that $\mathcal{T}_k$ is of sufficiently small mesh to contain a chain $\mathcal{C}$ as described in the statement of the theorem. If $\mathcal{T}_{k,1}$ and $\mathcal{T}_{k,2}$ are two coherent subcollections of $\mathcal{T}_k$ whose union is $\mathcal{T}_k$, then $\mathcal{C}_1 = \mathcal{C} \cap \mathcal{T}_{k,1}$ and $\mathcal{C}_2 = \mathcal{C} \cap \mathcal{T}_{k,2}$ will be two chains whose union is $\mathcal{C}$. By hypothesis one of these, say $\mathcal{C}_1$, spans $\mathcal{T}_n$. Thus $\mathcal{T}_{k,1}$ spans $\mathcal{T}_n$. Since the above is true for any n and any choice of $\mathcal{T}_{k,1}$ and $\mathcal{T}_{k,2}$, the indecomposability of X follows from the Cook-Ingram theorem (1.4).

4. Applications

Theorem 4.1. *Every star-like indecomposable continuum is almost chainable.*

Proof. Let X be an indecomposable star-like continuum and let $\mathcal{S}$ be an irreducible open star-cover of X. Let δ be as in theorem 3.9 and let $\mathcal{S}'$ be an irreducible star-cover refinement of $\mathcal{S}$ of mesh $\leq \delta$. Then $\mathcal{S}'$ contains a chain $\mathcal{C}$ as described in the previous theorem. Since $\mathcal{S}'$ is a star-cover,

$\mathcal{C}$ is contained in the union of two legs, $\mathcal{L}_1$ and $\mathcal{L}_2$, of $\mathcal{S}'$. It follows that either $\mathcal{L}_1$ or $\mathcal{L}_2$ spans $\mathcal{S}$. Suppose that $\mathcal{L}_1$ spans. Then $\mathcal{L}_1 \cup (\mathcal{S}' - \mathcal{L}_1)$ is an almost chain cover of X refining $\mathcal{S}$.

In [8] Jobe introduces the notion of a wide tree-like continuum and shows that these continua have the fixed point property. Jobe notes that all chainable continua are wide. We show below that these are in fact the only indecomposable wide tree-like continua. Further it is shown that chainable continua can be characterized as those continua which are atriodic wide tree-like.

Definition 4.2. A link B of a tree-chain $\mathcal{J}$ is called a *branch* link if it has non-void intersection with at least three other links of $\mathcal{J}$. If B is a branch link of $\mathcal{J}$ and $\mathcal{A}$ is a maximal coherent subcollection of $\mathcal{J}$-{B}, then $\mathcal{A}$ is called an arm of B.

Definition 4.3. A tree-like continuum X is said to be *wide* if X admits a defining sequence, $\mathcal{J}_1, \mathcal{J}_2, \cdots$, of open covers such that given any $\varepsilon > 0$ there exist a $\delta > 0$ and a natural number n such that if $k \geq n$, B is a branch link of $\mathcal{J}_k$ and $x \in X$ such that $d(x,B) > \varepsilon$; then $d(x,\cup\mathcal{A}) \geq \delta$ for any arm $\mathcal{A}$ of B which does not contain x.

Theorem 4.4. *Every indecomposable wide tree-like continuum is chainable.*

Proof. Let X be as above and let $\varepsilon > 0$ be given. Let δ and n be chosen for the number $\varepsilon/3$ as in the preceding definition. Now choose a $k \geq n$ such that $\mathcal{J}_k$ is of mesh $< \delta/2$

and let $k' > k$ be chosen such that $\mathcal{J}_{k'}$ is of mesh $< \varepsilon/3$ and contains a chain $\mathcal{C}'$ satisfying the condition of theorem 3.9. We may assume without loss of generality that $\mathcal{C}'$ is a maximal subchain of $\mathcal{J}_{k'}$.

Now let B be any branch link of $\mathcal{J}_{k'}$ which is an element of $\mathcal{C}'$. $\mathcal{C}'-\{B\}$ is the union of two chains $\mathcal{C}_1$ and $\mathcal{C}_2$ which are contained in arms $\mathcal{A}_1$ and $\mathcal{A}_2$ of B. Let x be a point of any other arm $\mathcal{A}$ of B. Since either $\mathcal{C}_1 \cup \{B\}$ or $\mathcal{C}_2 \cup \{B\}$ spans $\mathcal{J}_k$ and $\mathcal{J}_k$ is of mesh $< \delta/2$, either $d(x,\cup\mathcal{A}_1) < \delta$ or $d(x,\cup\mathcal{A}_2) < \delta$. It follows by hypothesis on δ and k' that $d(x,B) \leq \varepsilon/3$.

Thus every arm of B other than $\mathcal{A}_1$ and $\mathcal{A}_2$ can be amalgamated with B to form a new link B' of diameter $< \varepsilon$. Carrying out this construction on every branch link of $\mathcal{C}'$, every link of $\mathcal{J}_{k'}-\mathcal{C}'$ will get amalgamated with some link of $\mathcal{C}'$. Thus a new chain $\mathcal{C}$ is formed which covers X and has mesh $< \varepsilon$.

Theorem 4.4 can now be used together with a theorem of Fugate [6, Theorem 2] to obtain the following characterization of chainability.

Theorem 4.5. *A continuum is chainable if and only if it is atriodic wide tree-like.*

Proof. The necessity of the condition is obvious. If X is an atriodic wide tree-like continuum then it is hereditarily unicoherent. From Fugate's theorem it is sufficient to show that every indecomposable subcontinuum of X is chainable. Therefore let Y be an indecomposable subcontinuum of X. Then Y is also wide tree-like [8], and hence by Theorem 4.4 is chainable.

A second characterization of chainability can be obtained from the fact that wide tree-like continua of width zero are atriodic [8, corollary 1] (see [2] or [8] for a definition of width zero).

Theorem 4.6. *A continuum is chainable if and only if it is wide tree-like and of width zero.*

5. Remarks and Questions

Theorem 3.6 could have been stated as an "if and only if" theorem. One can see this by noting that it is only the conclusion of 3.6 which is used in place of indecomposability in the proofs of 3.7, 3.8 and 3.9. 3.6 is the analogue of 2.5 for tree-like continua, so one might guess that 2.5 would yield a characterization of indecomposability for all continua. It is not difficult to show that this is not the case. Any cover of the 2-cell can be refined with a cover containing a chain satisfying the Cook-Ingram condition in the first. Ray Russo, in a private communication with the authors, has produced a 1-dimensional example. The example consists of the one point union of two of Knaster's indecomposable continua with one endpoint (identified at the endpoints). This space can be regarded as a circularly chainable continuum if the opposite endlinks of the standard defining sequence of chains are amalgamated. This sequence of covers has the Cook-Ingram property, although the space is decomposable.

The work represented by this paper began from an attempt to prove the following corollary of 3.7:

Proposition 5.1. *Let* X *be an indecomposable tree-like* *continuum and let* $\mathcal{J}_1, \mathcal{J}_2, \cdots$ *be a defining sequence for* X. *Then given any* n, *there is an* m > n *such that for all* k $\geq$ m, $\mathcal{J}_k$ *contains a chain spanning* $\mathcal{J}_n$.

This result is also a corollary of results of Burgess [2]. However, 5.1 does not suffice to characterize indecomposability, since any almost chainable continuum satisfies the conclusion of 5.1. It would be interesting to know which indecomposable tree-like continua are almost chainable. Theorem 4.1 provides a partial answer. It appears from a conversation with D. Bellamy, A. Lelek and others at the Topology Conference that there is an example of an indecomposable tree-like continuum which is not almost chainable. It would be nice to have a simple example.

Question 5.2. Is every k-junctioned indecomposable tree-like continuum almost chainable?[4]

The answer to 5.2 is affirmative for homogeneous continua [1, Theorem 14].

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[4]Added in proof: The authors have answered this question in the negative with an example of a 2-junctioned indecomposable tree-like continuum which is not almost chainable.

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