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T. A. Chapman[1]

1. Introduction

Let M be a compact Q-manifold and let C(M) be the space of all continuous functions from M to M equipped with the sup norm metric. By the *space of retractions* of M we mean the closed subset R(M) of C(M) defined by $R(M) = \{e \in C(M) \mid e^2 = e\}$. The purpose of this paper is to prove the following result.

Theorem. R(M) *is an* ANR.

While it is well known that C(M) is an ANR, it is not at all obvious that R(M) is an ANR. In general, the problem of identifying subsets of C(M) which are ANRs is extremely difficult. Recently Ferry proved that the space of homeomorphisms of M is an ANR [4]. The main idea used there was the α-Approximation Theorem, which gave conditions under which a homotopy equivalence is close to a homeomorphism. This was then used to retract a neighborhood of a suitable function space onto the homeomorphism group of M.

Our strategy in the proof of the Theorem stated above is to find a neighborhood of R(M) in C(M) which retracts onto R(M). Note that the image of any element of R(M) is a compact ANR. So a first step in our proof is to show how each element of C(M) which is sufficiently close to R(M) gives rise to a compact ANR. Once this is done a retraction of M is constructed which has this ANR as its image. This then

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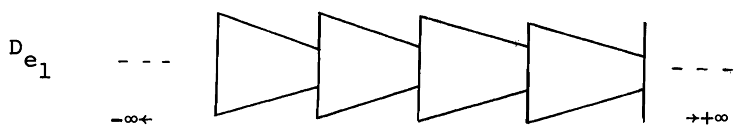
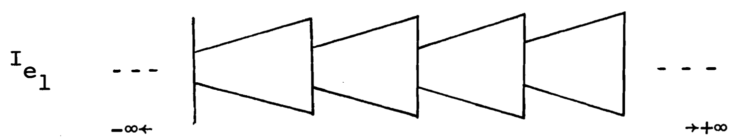
defines our map of a neighborhood of R(M) onto R(M). The hardest part of the proof is the construction of the compact ANR. To give the reader some idea of what is going on in the sequel we now give a brief description of how this is done.

Choose any $e \in C(M)$ which is close to R(M) and let $e_1: M \to M$ be a Z-embedding which is close to e. Then form the infinite direct mapping cylinder D_{e_1} and the infinite inverse mapping cylinder I_{e_1} as pictured below. They are formed by piecing together countably many copies of the mapping cylinder of $e_1: M \to M$ and both have natural projections to the reals R (see §2).





Since e_1 is a Z-embedding, both D_{e_1} and I_{e_1} are Q-manifolds. In §4 we construct a homeomorphism $h: I_{e_1} \to D_{e_1}$ which preserves the ends $+\infty$ and $-\infty$. By using retractions along mapping cylinder rays this enables us to get a strong deformation retraction of D_{e_1} to a compact Q-manifold. This is obtained by first deforming the end $-\infty$ in D_{e_1} along mapping cylinder rays of D_{e_1} to a compact portion of the space, and then deforming the end $+\infty$ in D_{e_1} along mapping cylinder rays supplied by $h(I_{e_1})$ to a compact portion of this space. The compact Q-manifold which is captured in between these two deformations is our desired compact ANR.

There is one part of this entire program which is quite unsatisfactory. In §§3, 4, and 5 there are "hidden conditions" which somehow say that if $e \in C(M)$ is sufficiently close to e^2, then e is close to a retraction. These conditions are unpleasant to formulate in one simple statement. In particular, they do not imply the following plausible looking question.

Question 1. *For every* $\varepsilon > 0$ *is there a* $\delta > 0$ *such that if* $e \in C(M)$ *and* $d(e,e^2) < \delta$, *then* e *is* ε*-close to a retraction?*

(Here d is the sup norm metric on C(M).)
This is related to the result of [3], which implies that if $e \in C(M)$ and if there is a pointed homotopy $e \simeq e^2$, then there is an obstruction in the projective class group $\tilde{K}_0\pi_1(M)$ to homotoping e to a retraction.

Since it is now fashionable to have ℓ_2-factors in function spaces (ℓ_2 = separable Hilbert space), the following question seems reasonable.

Question 2. *Is* R(M) *homeomorphic to* $R(M) \times \ell_2$*?*

By the results of [6], an affirmative answer would imply that R(M) is an ℓ_2-manifold.

In this paper we assume that the reader is familiar with basic Q-manifold apparatus such as can be found in the first four chapters of [2]. We also rely quite heavily on the paper

[4]. Some of our techniques of proof come from ideas in [4], and in some cases we directly use results which are established in [4].

2. Definitions and Notation

The purpose of this section is to introduce some material which will be used in the remaining sections. Recall from §1 that we are given a compact Q-manifold M which we will keep fixed throughout the remainder of this paper. We will regard M as a subset of the Hilbert cube Q and fix a neighborhood V of M for which there is a retraction of V onto M. If $x,y \in M$ are sufficiently close together, then the straight line segment $[x,y]$ from x to y lies in V, and therefore the retraction applied to $[x,y]$ defines a *canonical path* in M from x to y. Thus if $f,g\colon X \to M$ are maps which are sufficiently close, then there is a *canonical homotopy* from f to g which takes place along the canonical paths mentioned above. Now fix a neighborhood $U \subset C(M)$ of $R(M)$ such that if $e \in U$, then e and e^2 are so close that they are canonically homotopic. Denote this canonical homotopy by $e_t\colon e \simeq e^2$. Clearly if $e \in R(M)$, then $e = e^2$ and e_t is the constant homotopy.

For any map $f\colon X \to Y$ we let $M(f)$ denote its *mapping cylinder*. It is obtained from the disjoint union $X \times [0,1]$ $\amalg Y$ by identifying $(x,1)$ with $f(x)$. We write $M(f) = X \times [0,1) \cup Y$ and identify X with its 0-level, $X \times \{0\} \subset M(f)$. By the *rays* of $M(f)$ we mean the intervals $[x,1) \cup f(x) \subset M(f)$.

For any space X and map $f\colon X \to X$, the *infinite direct mapping cylinder* of f, denoted D_f, is the quotient space obtained from the disjoint union

$$\cdots \sqcup X \times [-1,0] \sqcup X \times [0,1] \sqcup X \times [1,2] \sqcup \cdots,$$

by identifying (x,n) in $X \times [n-1,n]$ with $(f(x),n)$ in $X \times [n,n+1]$. In a natural way D_f may be coordinatized by elements of $X \times R$.* Thus there is no confusion when we write $(x,t) \in D_f$. We will also identify $X \times \{t\}$ with an obvious subset of D_f, and in general we use $D_f[a,b]$ to denote the subset of D_f which corresponds to $X \times [a,b]$ in $X \times R$. The *infinite inverse mapping cylinder* of f, denoted I_f, is the quotient space obtained from the above disjoint union by identifying (x,n) in $X \times [n,n+1]$ with $(f(x),n)$ in $X \times [n-1,n]$. We also coordinatize I_f by $X \times R$ and use $I_f[a,b]$ to denote the subset of I_f which corresponds to $X \times [a,b]$.

We are going to need sliced versions of these direct and inverse mapping cylinder constructions. Let $E: M \times C(M) \to M \times C(M)$ be defined by $E(x,e) = (e(x),e)$ and form the infinite mapping cylinders I_E and D_E. Then I_E and D_E are both coordinatized by $M \times R \times C(M)$. Let $p: D_E \to C(M)$ be defined by $p(x,t,e) = e$ and let $q: I_E \to C(M)$ be defined by $q(x,t,e) = e$. Then $p^{-1}(e) = D_e$ and $q^{-1}(e) = I_e$. For each $e \in U$ define a map $u_e: D_e \to M \times R$ by $u_e(x,t) = (e_{t-n}(x),t)$, where $(x,t) \in D_e[n,n+1)$. Then define $u: p^{-1}(U) \to M \times R \times U$ by $u(x,t,e) = (u_e(x,t),e)$. The u_e's are the levels of u. Also define $d: M \times R \times U \to p^{-1}(U)$ by $d(x,t,e) = (d_e(x,t),e)$, where $d_e(x,t) = (e_{n+1-t}(x),t) \in D_e$, for $(x,t) \in D_e[n,n+1)$. Similarly, define $u': q^{-1}(U) \to M \times R \times U$ by $u'(x,t,e) = (u'_e(x,t),e)$, where $u'_e(x,t) = (e_{n+1-t}(x),t)$, for $(x,t) \in I_e(n,n+1]$. Finally, define $d': M \times R \times U \to q^{-1}(U)$ by $d'(x,t,e) = (d'_e(x,t),e)$,

*Be wary, this is a non-continuous coordinatization.

where $d'_e(x,t) = (e_{t-n}(x),t) \in I_e$, for $(x,t) \in M \times (n,n+1]$.

Now let $A \subset U$ and let $\pi: \mathcal{E} \to A$ be a continuous surjection. A map $f: \mathcal{E} \to p^{-1}(A)$ is *sliced* if $f\pi^{-1}(e) \subset p^{-1}(e)$, for all $e \in A$. We use $f_e: \pi^{-1}(e) \to p^{-1}(e)$ for the various levels of f. Two sliced maps $f,g: \mathcal{E} \to p^{-1}(A)$ are *strongly homotopic* if there is a sliced homotopy $\theta_t: f \simeq g$, a $k \geq 0$, and a map $\varepsilon: R(M) \cup A \to [0,\infty)$ such that

(1) ε is 0 on $R(M)$ and positive on $A - R(M)$,

(2) the levels $u_e(\theta_t)_e$, $u_e f_e: \pi^{-1}(e) \to M \times R$ differ by at most k in the R-coordinate and at most $\varepsilon(e)$ in the M-coordinate, for all e and t.

We say that f,g are *strongly close* if we omit the homotopy θ_t in the above definition by demanding that $u_e f_e, u_e g_e$: $\pi^{-1}(e) \to M \times R$ differ by at most k in the R-coordinate and at most $\varepsilon(e)$ in the M-coordinate. A sliced map $f: \mathcal{E} \to p^{-1}(A)$ is said to be a *strong homotopy equivalence* if there is a sliced map $g: p^{-1}(A) \to \mathcal{E}$ and sliced homotopies $\theta_t: fg \simeq id$, ϕ_t: $gf \simeq id$ such that θ_t and $f\phi_t$ are strong homotopies.

Write $Q = [0,1] \times [0,1] \times \cdots$ and let $s = (0,1) \times (0,1) \times \cdots$. Choose any homeomorphism $\alpha: M \to M \times Q$ and let $M_0 = \alpha^{-1}(M \times s)$. This gives us a *pseudo-interior* of M. Any compactum in M_0 is a Z-set in M, and there exist arbitrarily small maps of M into M_0. A very nice property of M_0 is that Z-sets in M_0 can be canonically unknotted in M [1]. Let $Z \subset C(M)$ be the space of embeddings whose images lie in M_0. Then each $e \in Z$ is a Z-embedding and clearly $Z \cap R(M) = \emptyset$. Note also that for $e \in Z$, I_e and D_e are Q-manifolds.

In the sequel we will be dealing with $p^{-1}(Z) \overset{p}{\to} Z$ and $q^{-1}(Z) \overset{q}{\to} Z$. The following useful observation tells us why these spaces are easier to handle than the full spaces $D_E \overset{p}{\to} C(M)$ and $I_E \overset{q}{\to} C(M)$.

Theorem 2.1. $p^{-1}(Z) \overset{p}{\to} Z$ *and* $q^{-1}(Z) \overset{q}{\to} Z$ *are locally trivial bundles.*

Proof. Just use the canonical Z-set unknotting results of [1].

3. Construction of a Strong Homotopy Equivalence

The main result of this section is Theorem 3.5, which proves that for some neighborhood $U_1 \subset U$ of $R(M)$ there is a strong homotopy equivalence of $q^{-1}(U_1)$ to $p^{-1}(U_1)$. For notation let $\Omega: D_E \to D_E$ be the sliced homeomorphism defined by $\Omega_e(x,t) = (x,t+1)$, and let $\Sigma: D_E \to D_E$ be the sliced map defined by $\Sigma_e(x,t) = (e(x),t)$. It is clear that by deforming down mapping cylinders in D_E (one notch to the right) we get a strong homotopy id $\simeq \Omega\Sigma$.

Lemma 3.1. *For some neighborhood* $U_1' \subset U$ *of* $R(M)$ *there is a strong homotopy* $\Sigma|p^{-1}(U_1') \simeq (\Sigma|p^{-1}(U_1'))^2$.

Proof. Choose U_1' so that for any $e \in U_1'$, the homotopy ee_t is very near the constant homotopy. Specifically, we want U_1' chosen so that the homotopies $ee_t: e^2 \simeq e^3$ and $e_te: e^2 \simeq e^3$ are canonically homotopic rel the ends. This means that there is a two-parameter homotopy $F_{t,u}: M \to M$ such that $F_{t,0} = ee_t$, $F_{t,1} = e_te$, and each $\{F_{t,u}(x) | 0 \leq u \leq 1\}$ is the canonical path from $ee_t(x)$ to $e_te(x)$. For any $e \in U_1'$ we now describe a homotopy of Σ_e to Σ_e^2. It will be clear from the construction that our desired strong homotopy from $\Sigma|p^{-1}(U_1')$ to $(\Sigma|p^{-1}(U_1'))^2$ can be obtained by applying these homotopies on the various levels.

It will suffice to construct a homotopy ϕ_t: $D_e[0,1] \to D_e[0,1]$ such that

(1) $\phi_0 = \Sigma_e | D_e[0,1]$,

(2) $\phi_1 = (\Sigma_e | D_e[0,1])^2$,

(3) $\phi_t | M \times \{0\}: M \times \{0\} \to M \times \{0\}$ is given by $\phi_t(x,0) = (e_t(x),0)$,

(4) $\phi_t | M \times \{1\}: M \times \{1\} \to M \times \{1\}$ is given by $\phi_t(x,1) = (e_t(x),1)$.

Conditions (3) and (4) define ϕ_t on the ends of $D_e[0,1]$. On $M \times \{1/2\}$ we obtain ϕ_t by deforming $(e(x),1/2)$ down the rays of $D_e[0,1]$ to the base, applying the homotopy ee_t: $e^2 \simeq e^3$, and then deforming back up $D_e[0,1]$ to $(e^2(x),1/2)$. On $M \times \{s\}$, $0 \leq s < 1/2$, ϕ_t is obtained by deforming $(e(x),s)$ to $(e(x),2s)$, applying e_t: $e \simeq e^2$, and then deforming back to $(e^2(x),s)$. Finally, on $M \times \{s\}$, $1/2 \leq s < 1$, ϕ_t is obtained as on $M \times \{1/2\}$, except that the homotopy $F_{t,2s-1}$: $e^2 \simeq e^3$ is used in the base.

Corollary 3.2. $\Sigma | p^{-1}(U_1')$ *is strongly homotopic to* id.

Proof. We have already observed the strong homotopy id $\simeq \Omega\Sigma$. Restricting to $p^{-1}(U_1')$ and multiplying on the right by Σ we get

$$\Sigma | p^{-1}(U_1') \simeq \Omega(\Sigma | p^{-1}(U_1'))^2 \simeq \Omega\Sigma | p^{-1}(U_1') \simeq \text{id}.$$

Definition. Define f.p. maps $f: q^{-1}(U) \to p^{-1}(U)$ and $g: p^{-1}(U) \to q^{-1}(U)$ by $f = du'$ and $g = d'u$.

Lemma 3.3. *For some neighborhood* $U_1 \subset U_1'$ *of* $R(M)$ *there is a strong homotopy* θ_t: $fg | p^{-1}(U_1) \simeq$ id.

Proof. Choose $U_1 \subset U_1'$ so that for each $e \in U_1$, the homotopy $e_{1-t}e_{1-t}e_te_t$: $e^6 \simeq e^6$ is canonically homotopic to the constant homotopy e^6: $e^6 \simeq e^6$ rel the ends. For any $e \in U_1$ we now describe a level of our homotopy θ_t. We want to describe a homotopy $f_eg_e \simeq \text{id}$ on D_e. We only need a homotopy $f_eg_e \simeq (\Sigma_e)^6$, for then Corollary 3.2 gives $(\Sigma_e)^6 \simeq \text{id}$.

To get $f_eg_e \simeq (\Sigma_e)^6$ consider $f_eg_e|D_e[0,1]$. On the ends it is given by $(x,0) \to (e^6(x),0)$ and $(x,1) \to (e^6(x),1)$. For any $(x,t) \in D_e[0,1], 0 \leq t < 1$, it is given by $(x,t) \to (e_{1-t}e_{1-t}e_te_t(x),t)$. Using our given canonical homotopy of $e_{1-t}e_{1-t}e_te_t$: $e^6 \simeq e^6$ to the constant homotopy we get a homotopy of $f_eg_e|D_e[0,1]$ to $(\Sigma_e)^6|D_e[0,1]$ rel the ends. This suffices to define our homotopy $f_eg_e \simeq (\Sigma_e)^6$.

Lemma 3.4. *There is a homotopy* ϕ_t: $gf|q^{-1}(U_1) \simeq \text{id}$ *such that* $f\phi_t$ *is a strong homotopy.*

Proof. Analogous to the proof of Lemma 3.3.

Theorem 3.5. $f|q^{-1}(U_1)$: $q^{-1}(U_1) \to p^{-1}(U_1)$ *is a strong homotopy equivalence.*

Proof. Apply Lemmas 3.3 and 3.4.

4. Construction of a Homeomorphism

Let f: $q^{-1}(U_1) \to p^{-1}(U_1)$ be the strong homotopy equivalence of Theorem 3.5. Our main result in this section is Theorem 4.5, where we prove that there is a neighborhood $U_2 \subset U_1$ of $R(M)$ and a sliced homeomorphism h: $q^{-1}(U_2 \cap Z) \to p^{-1}(U_2 \cap Z)$ which is strongly close to $f|q^{-1}(U_2 \cap Z)$.

It will be convenient to generalize some notation which was introduced in §2. Let $A \subset U$, π: $\mathcal{E} \to A$ be a surjection,

W be a metric space, and let $f,g\colon \mathcal{E} \to p^{-1}(A) \times W$ be sliced maps. This means that $f\pi^{-1}(e) \cup g\pi^{-1}(e) \subset p^{-1}(e) \times W$, for all $e \in A$. We say that f,g are *strongly close* if there is a $k \geq 0$ and a map $\varepsilon\colon R(M) \cup A \to [0,\infty)$ such that

(1) ε is 0 on $R(M)$ and positive on $A - R(M)$,

(2) the levels $(u_e \times id_W)f_e, (u_e \times id_W)g_e\colon \pi^{-1}(e) \to M \times R \times W$ differ by at most k in the R-coordinate and at most $\varepsilon(e)$ in the $M\times W$-coordinate, for all e.

(We use the standard product metric on $M\times W$.) Our procedure for constructing h is analogous to procedures used in [4]. The following is the first step.

Lemma 4.1. *There are sliced homeomorphisms*

$$h_1\colon q^{-1}(U_1 \cap Z) \times [-3,3) \to p^{-1}(U_1 \cap Z) \times [-3,3),$$
$$h_2\colon q^{-1}(U_1 \cap Z) \times (-3,3] \to p^{-1}(U_1 \cap Z) \times (-3,3]$$

which are strongly close to $f\times id$.

Proof. Using the fact that $p|p^{-1}(U_1 \cap Z)$ and $q|q^{-1}(U_1 \cap Z)$ are locally trivial bundles we proceed as in Proposition 5.5 of [4].

Definition. If h_1 and h_2 are constructed carefully, then $h_1h_2^{-1}(p^{-1}(U_1 \cap Z) \times (-2,2))$ lies in $p^{-1}(U_1 \cap Z) \times (-3,3)$. Thus

$$h = h_1h_2^{-1}|p^{-1}(U_1 \cap Z) \times (-2,2)\colon p^{-1}(U_1 \cap Z) \times (-2,2) \to p^{-1}(U_1 \cap Z) \times (-3,3)$$

defines an open embedding which is strongly close to id. In what follows we assume that $k = 1$ in the definition of h being strongly close to id. This does not detract from the generality of the argument. We also use h_e for the level of h taking $D_e \times (-2,2)$ into $D_e \times (-3,3)$.

The following is the main step in our proof of Theorem 4.5. It and Lemma 4.3 are technical results which are used to prove Lemma 4.4. Then Lemma 4.4 is used to prove Theorem 4.5.

Lemma 4.2. *For some neighborhood* $U_1' \subset U_1$ *of* R(M) *there is a sliced embedding*

$$\theta\colon (D_E[-5/2,5/2] \cap p^{-1}(U_1' \cap Z)) \times [-3/2,3/2] \to (D_E[-5/2,5/2] \cap p^{-1}(U_1' \cap Z)) \times (-3,3)$$

such that

(1) $\theta_e = \mathrm{id}$ *on* $(D_e\{-5/2\} \times [-3/2,3/2]) \cup (D_e\{5/2\} \times [-3/2,3/2])$, *for all* $e \in U_1' \cap Z$,

(2) $\theta_e = h_e$ *on* $D_e[-1/2,1/2] \times [-3/2,3/2]$, *for all* $e \in U_1' \cap Z$,

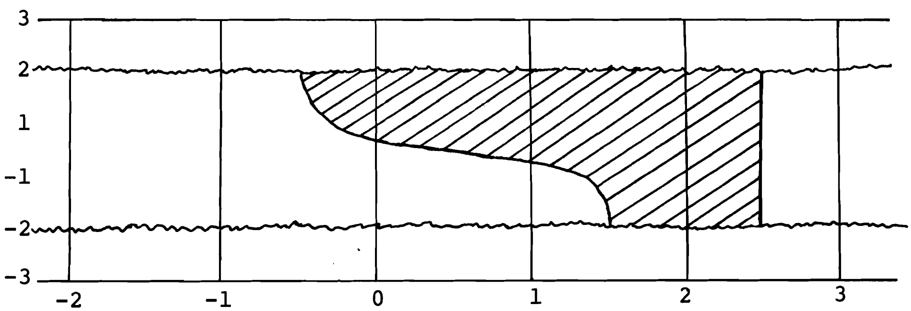
(3) the image of θ_e *contains* $D_e[-5/2,5/2] \times [-4/3,4/3]$, *for all* $e \in U_1' \cap Z$,

(4) θ *is strongly close to* id.

Proof. Define a subset X of $p^{-1}(U_1 \cap Z) \times (-3,3)$ as follows:

$$X = h((D_E[1/2,\infty) \cap p^{-1}(U_1 \cap Z)) \times (-2,2)) \cap ((D_E(-\infty,5/2] \cap p^{-1}(U_1 \cap Z)) \times (-3,3)).$$

The shaded region below is a picture of $X_e = X \cap (D_e \times (-3,3))$.



Let $r: X \to M \times \{5/2\} \times (U_1 \cap Z) \times (-3,3)$ be the sliced map arising from mapping cylinder collapses and let r_t: $X \to p^{-1}(U_1 \cap Z) \times (-3,3)$ be the sliced homotopy arising from mapping cylinder deformations such that $r_0 = \text{id}$ and $r_1 = r$. Also we let

$$\alpha: h((D_E[-3/2,1/2] \cap p^{-1}(U_1 \cap Z)) \times (-2,2)) \cup X \to X$$

be the retraction along mapping cylinder rays supplied by h. Then for some neighborhood $G_1 \subset U_1$ of $R(M)$,

$$r_t' = \alpha r_t: X \cap (p^{-1}(G_1 \cap Z) \times [-30/16,30/16]) \to X \cap (p^{-1}(G_1 \cap Z) \times (-3,3))$$

is a well-defined homotopy such that $r_0' = \text{id}$ and $r_1' = r|X \cap (p^{-1}(G_1 \cap Z) \times [-30/16,30/16])$. The remainder of the argument is carried out in the following five steps.

Step I. *The homotopy* $(rr_t')_e: X_e \cap (D_e \times [-30/16,30/16]) \to M \times \{5/2\} \times (-3,3)$ *gets "smaller" as* e *gets closer to* R(M), i.e. $\lim_{e \to R(M)} d((rr_t')_e, (rr_0')_e) = 0$.

Proof. Choose any $e \in G_1 \cap Z$ which is close to $R(M)$. We will prove that $(rr_t')_e$ is a small homotopy. There are two cases. Choose any point $(x,u,v) \in X_e \cap (D_e \times [-30/16, 30/16])$, where $(x,u) \in D_e$ and $v \in [-30/16,30/16]$. If $(r_t)_e(x,u,v) \in X_e$, then α has no effect and

$$(r\alpha r_t)_e(x,u,v) = (rr_t)_e(x,u,v) = r_e(x,u,v).$$

If $(r_t)_e(x,u,v) \notin X_e$, then we must have $u \leq 3/2$. This implies that $r_e(x,u,v) = (x_1,5/2,v)$, where x_1 is close to $e(x)$. We also have $(r_t)_e(x,u,v) = (x',u',v)$, where $e(x)$ is close to $e(x')$, and therefore $(\alpha r_t)_e(x,u,v) = (x'',u'',v')$, where $e(x'')$ is close to $e(x')$ and v' is close to v. Finally, $(r\alpha r_t)_e(x,u,v) = (x''',5/2,v')$, where x''' is close to $e(x'')$.

All this means that x''' is close to x_1.

Remark. We have just shown that $r|X \cap (p^{-1}(G_1 \cap Z) \times (-3,3))$ is a "small equivalence" over $M \times \{5/2\} \times (G_1 \cap Z) \times [-30/16,30/16]$, for G_1 close to R(M). Such maps can be locally converted to homeomorphisms (see Theorem 3.6 of [4]). We will need this in the next step.

Step II. *There is a neighborhood* $G_2 \subset G_1$ *of* R(M) *and a subset* Y *of* $X \cap (p^{-1}(G_2 \cap Z) \times (-3,3))$ *so that*

(1) Y *contains* $X \cap (p^{-1}(G_2 \cap Z) \times [-7/4,7/4])$,

(2) there exists a sliced homeomorphism $\phi: Y \to M \times \{5/2\} \times (G_2 \cap Z) \times [-29/16,29/16]$ *which is strongly close to* $r|Y$,

(3) $\phi|M \times \{5/2\} \times (G_2 \cap Z) \times [-3/2,3/2] = \text{id}$.

Proof. Let

$$r' = r|X \cap (p^{-1}(G_1 \cap Z) \times (-3,3)): X \cap (p^{-1}(G_1 \cap Z) \times (-3,3)) \to M \times \{5/2\} \times (G_1 \cap Z) \times (-3,3).$$

It follows from Step I that there is a map $\varepsilon: G_1 \cap Z \to (0,\infty)$ which extends to a map $\tilde{\varepsilon}: R(M) \cup (G_1 \cap Z) \to [0,\infty)$ satisfying $\tilde{\varepsilon}(R(M)) = 0$, and r' is a sliced ε-equivalence over $M \times \{5/2\} \times (G_1 \cap Z) \times [-30/16,30/16]$. (See Definition 3.5 of [4].) In analogy with Theorem 3.6 of [4] there is a neighborhood $G_2 \subset G_1$ of R(M) and a subset Y of $X \cap (p^{-1}(G_2 \cap Z) \times (-3,3))$ so that

(1) Y contains $X \cap (p^{-1}(G_2 \cap Z) \times [-7/4,7/4])$,

(2) there exists a sliced homeomorphism $\phi': Y \to M \times \{5/2\} \times (G_2 \cap Z) \times [-29/16,29/16]$ which is sliced δ-homotopic to $r'|Y$,

where $\delta: G_2 \cap Z \to (0,\infty)$ is a map which extends to a map

$\tilde{\delta}$: $R(M) \cup (G_2 \cap Z) \to [0,\infty)$ satisfying $\tilde{\delta}(R(M)) = \{0\}$.

Note that (1) implies that $M \times \{5/2\} \times (G_2 \cap Z) \times [-3/2,3/2]$ is a sliced Z-set in Y. (See §4 of [4] for an excellent presentation of sliced Z-sets.) By (2) we note that $\phi'|M \times \{5/2\} \times (G_2 \cap Z) \times [-3/2,3/2]$ a sliced Z-embedding which is sliced δ-homotopic to id. Thus by Theorem 4.4 of [4] we can correct ϕ' to obtain a sliced homeomorphism $\phi: Y \to M \times \{5/2\} \times (G_2 \cap Z) \times [-29/16,29/16]$ which is δ-homotopic to ϕ' and which satisfies $\phi|M \times \{5/2\} \times (G_2 \cap Z) \times [-3/2,3/2] = \text{id}$. It is easy to check that ϕ is strongly close to $r|Y$.

Step III. *There is a sliced homeomorphism* ψ: $(D_E[1/2,5/2] \cap p^{-1}(G_2 \cap Z)) \times [-29/16,29/16] \to Y$ *such that*

(1) $\psi = h$ *on* $M \times \{1/2\} \times (G_2 \cap Z) \times [-3/2,3/2]$,

(2) $\psi = \text{id}$ *on* $M \times \{5/2\} \times (G_2 \cap Z) \times [-3/2,3/2]$,

(3) ψ *is strongly close to* id.

Proof. Consider a composition of sliced homeomorphisms,

$$\psi': (D_E[1/2,5/2] \cap p^{-1}(G_2 \cap Z)) \times [-29/16,29/16] \to M \times \{5/2\} \times (G_2 \cap Z) \times [-29/16,29/16] \xrightarrow{\phi^{-1}} Y,$$

where the first homeomorphism is obtained from mapping cylinder collapses. By Theorem 4.4 of [4] we can correct ψ' to obtain our desired ψ.

Step IV. *There is a sliced embedding*

$$\psi_1: (D_E[1/2,5/2] \cap p^{-1}(G_2 \cap Z)) \times [-3/2,3/2] \to X \cap (p^{-1}(G_2 \cap Z) \times (-3,3))$$

such that

(1) $\psi_1 = h$ *on* $M \times \{1/2\} \times (G_2 \cap Z) \times [-3/2,3/2]$,

(2) $\psi_1 = \text{id}$ *on* $M \times \{5/2\} \times (G_2 \cap Z) \times [-3/2,3/2]$,

(3) ψ_1 *is strongly close to* id,

(4) the image of ψ_1 *contains* $X \cap (p^{-1}(G_2 \cap Z) \times [-4/3, 4/3])$,

(5) $\psi_1^{-1}(M \times \{5/2\} \times (G_2 \cap Z) \times (-3,3)) = M \times \{5/2\} \times (G_2 \cap Z) \times [-3/2,3/2]$,

(6) $\psi_1^{-1}(h(M \times \{1/2\} \times (G_2 \cap Z) \times (-2,2))) = M \times \{1/2\} \times (G_2 \cap Z) \times [-3/2,3/2]$.

Proof. Let $\psi_1' = \psi | D_E[1/2,5/2] \cap p^{-1}(G_2 \cap Z) \times [-3/2,3/2]$. Then ψ_1' satisfies (1)-(4). To get an embedding ψ_1 which satisfies (5) and (6) we just push the image of ψ_1' away from the ends.

Step V. *There is a sliced embedding*

$$\psi_2\colon (D_E[-5/2,-1/2] \cap p^{-1}(G_2 \cap Z)) \times [-3/2,3/2] \to X' \cap (p^{-1}(G_2 \cap Z) \times (-3,3)),$$

where

$$X' = h((D_E(-\infty,-1/2] \cap p^{-1}(U_1 \cap Z)) \times (-2,2)) \cap ((D_E[-5/2,\infty) \cap p^{-1}(U_1 \cap Z)) \times (-3,3)),$$

such that

(1) $\psi_2 = \text{id}$ *on* $M \times \{-5/2\} \times (G_2 \cap Z) \times [-3/2,3/2]$,

(2) $\psi_2 = h$ *on* $M \times \{-1/2\} \times (G_2 \cap Z) \times [-3/2,3/2]$,

(3) ψ_2 *is strongly close to* id,

(4) the image of ψ_2 *contains* $X' \cap (p^{-1}(G_2 \cap Z) \times [-4/3,4/3])$,

(5) $\psi_2^{-1}(M \times \{-5/2\} \times (G_2 \cap Z) \times (-3,3)) = M \times \{-5/2\} \times (G_2 \cap Z) \times [-3/2,3/2]$,

(6) $\psi_2^{-1}(h(M \times \{-1/2\} \times (G_2 \cap Z) \times (-2,2)) = M \times \{-1/2\} \times (G_2 \cap Z) \times [-3/2,3/2]$.

Proof. Similar to Step IV.

We now show how to finish off the proof. Let $U_1' = G_2$; define θ by piecing together ψ_2 on $(D_E[-5/2,-1/2] \cap p^{-1}(U_1' \cap Z)) \times [-3/2,3/2]$, ψ_1 on $(D_E[1/2,5/2] \cap p^{-1}(U_1' \cap Z)) \times [-3/2,3/2]$, and h on $(D_E[-1/2,1/2] \cap p^{-1}(U_1' \cap Z)) \times [-3/2, 3/2]$. By construction, θ fulfills our requirements.

Lemma 4.3. *There is a neighborhood* $U_1'' \subset U_1'$ *of* R(M) *and a sliced homeomorphism* g_1: $p^{-1}(U_1'' \cap Z) \times (-3,3) \to p^{-1}(U_1'' \cap Z) \times (-3,3)$ *such that*

(1) g_1 *is supported on* $p^{-1}(U_1'' \cap Z) \times [-2,2]$

(2) g_1 *is strongly close to* id,

(3) $g_1 = h$ *on* $(D_E[7n-1/2,7n+1/2] \cap p^{-1}(U_1'' \cap Z)) \times [-1,1]$, *for* n *any integer.*

Proof. We will construct a sliced homeomorphism α of $(D_E[-7/2,7/2] \cap p^{-1}(U_1'' \cap Z)) \times (-3,3)$ onto itself such that

(1) $\alpha = \text{id}$ on $M \times (U_1'' \cap Z) \times \{-7/2,7/2\} \times (-3,3)$,

(2) α is supported on $(D_E[-7/2,7/2] \cap p^{-1}(U_1'' \cap Z)) \times [-2,2]$,

(3) $\alpha = h$ on $(D_E[-1/2,1/2] \cap p^{-1}(U_1'' \cap Z)) \times [-1,1]$,

(4) α is strongly close to id.

By patching together an infinite number of such homeomorphisms we can easily obtain our desired g_1. So all we have to do is construct α.

Let θ be the sliced embedding of Lemma 4.2. Extend θ via the identity to a sliced embedding

$$\tilde{\theta}: (D_E[-7/2,7/2] \cap p^{-1}(U_1' \cap Z)) \times [-3/2,3/2] \to (D_E[-7/2,7/2] \cap p^{-1}(U_1' \cap Z)) \times (-3,3).$$

Let ϕ: $(D_E[-7/2,7/2] \cap p^{-1}(U_1' \cap Z)) \times (-3,3) \to M \times \{7/2\} \times (U_1' \cap Z) \times (-3,3)$ be a sliced homeomorphism obtained from mapping cylinder collapses, and consider the embedding

$$\lambda: M \times \{7/2\} \times (U_1' \cap Z) \times [-3/2,3/2]$$
$$\overset{\phi^{-1}}{\to} (D_E[-7/2,7/2] \cap p^{-1}(U_1' \cap Z)) \times [-3/2,3/2]$$
$$\overset{\tilde{\theta}}{\to} (D_E[-7/2,7/2] \cap p^{-1}(U_1 \cap Z)) \times (-3,3)$$
$$\overset{\phi}{\to} M \times \{7/2\} \times (U_1' \cap Z) \times (-3,3).$$

(Certainly ϕ can be constructed so that the (-3,3)-coordinates are not affected.) Note that the image of λ contains $M \times \{7/2\} \times (U_1' \cap Z) \times [-4/3,4/3]$. Thus $\lambda|M \times \{7/2\} \times (U_1' \cap Z) \times (-5/4,5/4)$ is an open embedding. By the Deformation Theorem of [5] we can choose a neighborhood $U_1'' \subset U_1'$ of $R(M)$ and a sliced homeomorphism α_1 of $M \times \{7/2\} \times (U_1'' \cap Z) \times (-3,3)$ onto itself such that

(1) α_1 is supported on $M \times \{7/2\} \times (U_1'' \cap Z) \times [-2,2]$,

(2) α_1 = id on $\phi(M \times \{-7/2,7/2\} \times (U_1'' \cap Z) \times (-3,3))$,

(3) $\alpha_1 = \lambda$ on $M \times \{7/2\} \times (U_1'' \cap Z) \times [-1,1]$,

(4) α_1 is strongly close to id.

Then $\alpha = \phi^{-1}\alpha_1\phi$ fulfills our requirements.

Lemma 4.4. *There is a neighborhood* $U_2 \subset U_1$ *of* $R(M)$ *and a sliced homeomorphism* $\tilde{h}: p^{-1}(U_2 \cap Z) \times (-3,3) \to p^{-1}(U_2 \cap Z) \times (-3,3)$ *such that*

(1) $\tilde{h}$ *is supported on* $p^{-1}(U_2 \cap Z) \times [-2,2]$,

(2) $\tilde{h} = h$ *on* $p^{-1}(U_2 \cap Z) \times [-1/2,1/2]$,

(3) $\tilde{h}$ *is strongly close to* id.

Proof. If g_1 is as in Lemma 4.3, consider the open embedding

$$h_1 = g_1^{-1}h: p^{-1}(U_1'' \cap Z) \times (-2,2) \to p^{-1}(U_1'' \cap Z) \times (-3,3).$$

We have h_1 = id on $(D_E[7n-1/2,7n+1/2] \cap p^{-1}(U_1'' \cap Z)) \times [-1,1]$

and h_1 is strongly close to id. If we again use the ideas of Lemma 4.3 we can find a sliced homeomorphism $\tilde{g}_2$ of $(D_E[1/2,7+1/2] \cap p^{-1}(U_2 \cap Z)) \times (-3,3)$ onto itself (for some neighborhood $U_2 \subset U_1''$ of $R(M)$) such that

(1) $\tilde{g}_2$ is supported on $(D_E[1/2,7+1/2] \cap p^{-1}(U_2 \cap Z)) \times [-2,2]$,

(2) $\tilde{g}_2$ is strongly close to id,

(3) $\tilde{g}_2 = h_1$ on $(D_E[1/2,7+1/2] \cap p^{-1}(U_2 \cap Z)) \times [-1/2, 1/2]$,

(4) $\tilde{g}_2 = \text{id}$ on $M \times \{1/2,7+1/2\} \times (U_2 \cap Z) \times (-3,3)$.

Then homeomorphisms of this type piece together to give a homeomorphism g_2 of $p^{-1}(U_2 \cap Z) \times (-3,3)$ onto itself. Putting $\tilde{h} = g_1g_2$ we are done.

Theorem 4.5. *There is a neighborhood* $U_2 \subset U_1$ *of* $R(M)$ *and a sliced homeomorphism* $\hat{h}: q^{-1}(U_2 \cap Z) \to p^{-1}(U_2 \cap Z)$ *which is strongly close to* $f|q^{-1}(U_2 \cap Z)$.

Proof. Let $\tilde{h}$ be as in Lemma 4.4. Then

$$h_1|q^{-1}(U_2 \cap Z) \times [-3,1/2]: q^{-1}(U_2 \cap Z) \times [-3,1/2] \to p^{-1}(U_2 \cap Z) \times [-3,3],$$

$$\tilde{h}h_2|q^{-1}(U_2 \cap Z) \times [-1/2,3]: q^{-1}(U_2 \cap Z) \times [-1/2,3] \to p^{-1}(U_2 \cap Z) \times [-3,3]$$

are sliced embeddings which piece together to give a sliced homeomorphism

$$h': q^{-1}(U_2 \cap Z) \times [-3,3] \to p^{-1}(U_2 \cap Z) \times [-3,3]$$

which is strongly close to $f\times\text{id}$. Since $q^{-1}(U_2 \cap Z) \xrightarrow{q} U_2 \cap Z$ and $p^{-1}(U_2 \cap Z) \xrightarrow{p} U_2 \cap Z$ are locally trivial bundles we can push in the $[-3,3]$-factor to obtain our desired result.

5. Construction of a Retraction

The main result of this section is Theorem 5.2, where we prove that there is a neighborhood $G \subset U$ of $R(M)$ and a retraction of $R(M) \cup (G \cap Z)$ to $R(M)$.

Let $u: p^{-1}(U) \to M \times R \times U$ be the map of §2 and consider the restriction $u: p^{-1}(U \cap Z) \to M \times R \times (U \cap Z)$. We know that $p^{-1}(U \cap Z) \overset{p}{\to} U \cap Z$ is a locally trivial bundle. It is easy to approximate u by a sliced Z-embedding $\tilde{u}: p^{-1}(U \cap Z) \to M \times R \times (U \cap Z)$. This can be done so that $\tilde{u}$ is *strongly homotopic* to u, i.e. there exists a sliced homotopy $u_t: \tilde{u} \simeq u$, a $k \geq 0$, and a map $\varepsilon: R(M) \cup (U \cap Z) \to [0,\infty)$ such that

(1) ε is 0 on $R(M)$ and positive on $U \cap Z$,

(2) the levels $(u_t)_e, u_e: D_e \to M \times R$ differ by at most k in the R-coordinate and at most $\varepsilon(e)$ in the M-coordinate, for all e and t.

This should be compared (but not confused) with the definition given in §2 of strongly homotopic maps into D_E. Here is a result which will be needed in the proof of Theorem 5.2.

Lemma 5.1. *There is a neighborhood* $G_1 \subset U$ *of* $R(M)$ *and a sliced retraction* $s: M \times R \times (G_1 \cap Z) \to \tilde{u}p^{-1}(G_1 \cap Z)$ *which is strongly close to the map* Σ *of* $M \times R \times (G_1 \cap Z)$ *to itself which sends* (x,t,e) *to* $(e(x),t,e)$.

Proof. Consider the map $s' = \tilde{u}d: M \times R \times (U \cap Z) \to \tilde{u}p^{-1}(U \cap Z)$ and let $G_1 \subset U$ be a neighborhood of $R(M)$ so that $du: p^{-1}(G_1) \to p^{-1}(G_1)$ is strongly homotopic to id (compare with Lemma 3.3). We will homotop $s'|M \times R \times (G_1 \cap Z)$ to our desired sliced retraction.

Note that $s'|\tilde{u}p^{-1}(G_1 \cap Z)$ is given by $\tilde{u}d$, which factors into $\tilde{u}d\tilde{u}\tilde{u}^{-1}$. Since $\tilde{u}$ is strongly homotopic to u and du is strongly homotopic to id, we conclude that $s'|\tilde{u}p^{-1}(G_1 \cap Z)$ is strongly homotopic to id with the homotopy taking place in $\tilde{u}p^{-1}(G_1 \cap Z)$. Consider the subset

$$X = (M \times R \times (G_1 \cap Z) \times \{0\}) \cup (\tilde{u}p^{-1}(G_1 \cap Z) \times I)$$

of $M \times R \times (G_1 \cap Z) \times I$. It is a sliced Z-set and therefore it can be fiberwise collared (see Corollary 4.10 of [4]). This means that we can find a neighborhood $W \subset M \times R \times (G_1 \cap Z) \times I$ of X and a sliced retraction $\theta_1: W \to X$. Define $\theta_2: M \times R \times (G_1 \cap Z) \times I \to W$ by $\theta_2(\alpha,t) = (\alpha,\phi(\alpha)\cdot t)$, where $\phi: M \times R \times (G_1 \cap Z) \to [0,1]$ is a map which takes $\tilde{u}p^{-1}(G_1 \cap Z)$ to 1.* Define $h: X \to \tilde{u}p^{-1}(G_1 \cap Z)$ by $h(\alpha,0) = s'(\alpha)$, for $\alpha \in M \times R \times (G_1 \cap Z)$, and $h|\tilde{u}p^{-1}(G_1 \cap Z) \times I$ is given by the strong homotopy $s'|\tilde{u}p^{-1}(G_1 \cap Z) \simeq \text{id}$. Then our desired $s: M \times R \times (G_1 \cap Z) \to \tilde{u}p^{-1}(G_1 \cap Z)$ is defined by $s(\alpha) = h\theta_1\theta_2(\alpha,1)$.

Now let $\hat{h}: q^{-1}(U_2 \cap Z) \to p^{-1}(U_2 \cap Z)$ be the sliced homeomorphism of §4 and let $G = G_1 \cap U_2$. Then $\tilde{u}\hat{h}|q^{-1}(G \cap Z)$ is strongly close to $u': q^{-1}(G \cap Z) \to M \times R \times (G \cap Z)$.

Theorem 5.2. *There is a retraction of* $R(M) \cup (G \cap Z)$ *to* $R(M)$.

Proof. We will define a map $\theta: G \cap Z$ to $R(M)$ which extends to our desired retraction. To simplify notation we assume that for any $(x,t,e) \in q^{-1}(G \cap Z)$ and $\hat{h}(x,t,e) = (x',t',e)$, $|t-t'| \leq 1$. With this in mind define

$$N = (D_E[-1,\infty) \cap p^{-1}(G \cap Z)) - \hat{h}(I_E(1,\infty) \cap q^{-1}(G \cap Z)).$$

---

*We are imitating here the usual proof of the homotopy extension theorem.

It is not hard to see that $p: N \to G \cap Z$ is a locally trivial bundle with compact Q-manifold fiber which lies in $D_E[-1,2] \cap p^{-1}(G \cap Z)$.

We can construct a sliced retraction $\theta_1: D_E[-4,5] \cap p^{-1}(G \cap Z) \to N$ which is strongly close to id by

(1) using the mapping cylinder retraction of $D_E[-4,-1] \cap p^{-1}(G \cap Z)$ to $M \times R \times \{-1\} \times (G \cap Z)$,

(2) use the mapping cylinder retraction of $I_E[1,\infty) \cap q^{-1}(G \cap Z)$ to $M \times R \times \{1\} \times (G \cap Z)$ (conjugated with $\hat{h}$).

Then $\theta_2 = \tilde{u}\theta_1\tilde{u}^{-1}$ is a sliced retraction of $\tilde{u}(D_E[-4,5] \cap p^{-1}(G \cap Z))$ to $\tilde{u}(N)$ which is strongly close to id.

Recall the sliced retraction $s: M \times R \times (G \cap Z) \to \tilde{u}p^{-1}(G \cap Z)$ of Lemma 5.1 which is strongly close to Σ. For the sake of simplicity assume that the bound on s in the R-coordinate is 1. Then we must have

$$s(M \times [-2,3] \times (G \cap Z)) \subset \tilde{u}(D_E[-4,5] \cap p^{-1}(G \cap Z)).$$

Also we have $\tilde{u}(N) \subset M \times [-2,3] \cap (G \cap Z)$. Thus $\theta_3 = \theta_2 s$ defines a retraction of $M \times [-2,3] \times (G \cap Z)$ to $\tilde{u}(N)$ which is strongly close to Σ.

Let $k: M \times [-2,3] \times (G \cap Z) \to M \times (G \cap Z)$ be a sliced homeomorphism so that as $e \in G \cap Z$ gets closer to $R(M)$, $k_e: M \times [-2,3] \to M$ gets closer to projection. Then our desired $\theta: G \cap Z \to R(M)$ is defined by letting $\theta_e: M \to M$ be given by $\theta_e = k_e(\theta_3)_e k_e^{-1}$. It is easy to check that θ fulfills our requirements.

6. Proof of the Theorem

By use of Theorem 5.2 the proof is fairly straightforward.

Any map e: $M \to M$ can be approximated by a Z-embedding into M_0, so it is clear that there is a map ϕ: $C(M) \to R(M) \cup Z$ which is the identity on $R(M)$. If θ: $R(M) \cup (G \cap Z) \to R(M)$ is the retraction of Theorem 5.2, then $\theta\phi$: $W \to R(M)$ is a retraction, where $W \subset C(M)$ is a neighborhood of $R(M)$ which lies in $\phi^{-1}(R(M) \cup (G \cap Z))$.

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